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— 13 —

Tetrahedron Geometric Factors

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13.1 Introduction

The geometrical shape of the tetrahedron formed by the spacecraft is an essential criterion in the choice of scientific investigation which can be performed with data from a multi-spacecraft mission, such as Cluster. The shape of this polyhedron evolves continuously along the orbital trajectory of the spacecraft, and this shape has a major impact on the accuracy of the determination of scientific parameters related to the spatial gradient, such as the current density, which is discussed in Chapter 16. The scientific importance of the shape, combined with its variability, inevitably led to many proposals for "quality factors" to attempt to describe the geometric shape of the tetrahedron, or for "performance indicators", to indicate the likely error of a particular scientific parameter.

While these early geometric factors were all one-dimensional, 2-D parameters to characterise the geometrical shape of the tetrahedron have also been proposed, i.e., the "elongation" E and "planarity" P defined in terms of the eigenvalues of the volumetric tensor described in Chapter 12.

In this chapter, we use the E and P parameters to define five characteristic types of tetrahedra and we check the validity and the meaning of the 1-D geometric factors by a numerical simulation using an "homogeneous reservoir of tetrahedra" in the E-P configuration space. As a practical application, we present an example of the Cluster orbit, and the associated computation of the 1-D and 2-D geometric factors. We represent these quantities in the E-P diagram, which allows a better understanding of their meaning. Finally, we demonstrate the limits of the 1-D geometric factors and point out the advantages of a 2-D geometric factor.

13.2 Measurement Performance

Tetrahedral geometry is one of the principal factors affecting measurement performance; that is the precision of physical parameters derived by comparison of data acquired at four points in space. There are, in fact, three factors which affect this precision: the tetrahedral geometry, the structure (in time and space) of the phenomena sampled, and the inevitable experimental errors inherent to all physical measurements. Measurement accuracy includes not only instrumental accuracy, but also timing and location accuracy. As mentioned in the introduction, the treatment of errors is covered by Chapters 11, 16, 17, in terms of the determination of different sets of criteria for each physical parameter to be determined. Different analysis techniques are appplied to four-point measurements to derive different physical parameters, such as the local current density (involving spatial field gradients), wave vector or mode, or global structure (boundaries). Each technique imposes different criteria on adequate sampling for measurement quality. Each of these criteria could be monitored separately, or given differing emphasis, depending on which particular physical property is of interest.

Measurement quality is therefore not determined only by the geometric "quality" of the tetrahedron (or polyhedron). Even for events which do not evolve structurally with time, the sampling achieved *of the physical event* depends upon the geometry (and scale) of the tetrahedron relative to that of the physical structure (anisotropy of the phenomena) present. For a highly anisotropic physical structure, a particular alignment of an anisotropic tetrahedral spacecraft configuration may be optimal, for example, to determine the spatial gradient. Different relative event scales, however, will result in different measurement performance for any given tetrahedral size and shape. Multipoint analysis typically involves the determination of gradients so that, for any given polyhedron overall size, derived quantities will, typically, be sensitive to the tetrahedral geometry when sampling similar physical structures.

Measurement quality depends also on the size of the tetrahedron, compared to the product of the measurement time resolution and the spacecraft relative velocity with respect to the physical structure, i.e. the interval the spacecraft travel into the structure within one data accumulation period. Note that for the particle experiments the data accumulation period is typically equal to the spacecraft spin, so as to sample a complete 3-D distribution function. In the case of Cluster this is 4 seconds, which determines the minimum size the tetrahedron should have for the various spacecraft/physical-structure relative velocities.

Consider Figure 13.1, for example, which indicates the evolution of the spacecraft configuration around a Cluster orbit for two proposed scenarios. Note how very different the evolution is and how the geometry varies widely in shape and size over the orbit. The insets show enlarged (by a factor of 50 with respect to the main figure) configurations, projected into the plane of view. The first group of insets, at positions 1, 2, and 3, show a highly elongated configuration at the southern magnetopause crossing (3). The orientation of this can be changed (not simply) by changing the orientation at 1 and, for instance, for some simulated mission phases has a more parallel alignment to the boundary. Faced with such a predicted tetrahedral geometry, the physical parameters which can be well determined depend upon the orientation with respect to the boundary. For such a nearly 1-D structure, techniques which determine those parameters depending on spatial structure (gradients, such as for $\nabla \times \boldsymbol{B}$ or $\nabla \times \boldsymbol{V}$) will typically require a configuration aligned with the boundary (i.e., matching the small and large gradients), but techniques which analyse

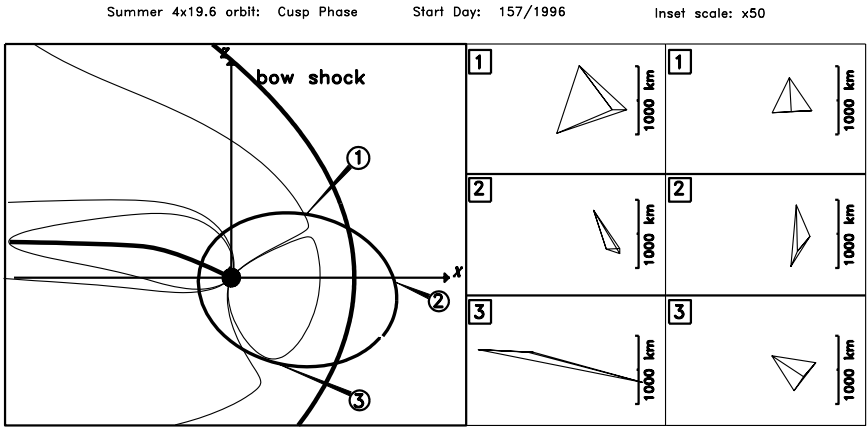

Figure 13.1: Evolution of the Cluster configuration around the nominal orbit for the day-side phase. The insets show two options which both target the northern cusp with a regular tetrahedron, but target the southern cusp with a regular tetrahedron only for the second case. The first option is optimised for fuel. [Reproduced from *Balogh et al.*, 1997.]

macroscopic properties will benefit from an anisotropic configuration in different ways: motional properties will be best sampled by perpendicular alignment, whereas boundary shape (especially non-planar) is best sampled by parallel alignment.

A quality parameter that monitors only spacecraft configuration is particularly useful, however, when sampling of structure is not important. Such a parameter would best reflect performance relating to transient or fluctuating events, for instance, with no preferred orientation to the global structure. A large number of events are not predictable and therefore a regular tetrahedron is optimum in this situation. We call here a regular tetrahedron a particular tetrahedron where the separations between each pair of points are equal. For the second scenario in the figure, for instance, the target at 3 has been chosen to correspond to a second, regular tetrahedron in an attempt to regulate the evolution over the orbit. The effect of tetrahedral distortion in terms of geometric quality parameters is studied in detail in the next section.

Optimum configurations in terms of either physical sampling or measurement uncertainty, as discussed above, are only likely to be achieved over small segments of the orbit. It would seem sensible, therefore, to attempt to optimise for data quality over selected global regions, together with choice of spatial scale, as a primary constraint. For other regions, use may be made of the natural distortion of the configuration to achieve preferred orientations with respect to the sampled structure.

13.3 The Shape of the Tetrahedron

13.3.1 The 1-Dimensional Geometric Factors

Four points in space define a *tetrahedron*. If the separations between each pair of points are equal, then it is a *regular* tetrahedron. Four spacecraft will form a tetrahedron,

which in general will not be regular. How can we specify the degree to which regularity is achieved? A number of parameters have been proposed to accomplish this, which we present and compare below.

The Q_{GM} Parameter

The Q_{GM} parameter is defined as

$$Q_{GM} = \frac{\text{True Volume}}{\text{Ideal Volume}} + \frac{\text{True Surface}}{\text{Ideal Surface}} + 1 \tag{13.1}$$

The *ideal* volume and surface are calculated for a regular tetrahedron with a side length equal to the average of the 6 distances between the 4 points.

Q_{GM} takes values between 1 and 3, and attempts to describe the "fractional dimension" of the tetrahedron: a value of 1 indicates that the four spacecraft are in a line, while a value equal to 3 indicates that the tetrahedron is regular. There is nevertheless some difficulty with this interpretation: it is perfectly possible to deform a regular ($Q_{GM} = 3$) tetrahedron continuously until it resembles a straight line ($Q_{GM} = 1$) without it resembling a plane at any time; therefore $Q_{GM} = 2$ is not a sufficient condition for planarity.

The Q_{RR} Parameter

The Q_{RR} parameter is defined to be

$$Q_{RR} = \left(\frac{9\pi}{2\sqrt{3}} \cdot \frac{\text{True Volume}}{\text{Sphere Volume}} \right)^{\frac{1}{3}} \tag{13.2}$$

where the sphere is that circumscribing the tetrahedron (all four points on its surface). Q_{RR} is normalised to be equal to 1 for a regular tetrahedron; its minimum value is 0. This parameter was selected from many on the basis of its usefulness in estimating the error in the determination of the spatial gradient of the magnetic field. This is discussed in section 13.5.4.

The Q_{SR} Geometric Factor

Another of the 1-D parameters is known as the Q_{SR} geometric factor. This factor is simply defined by:

$$Q_{SR} = \frac{1}{2} \left(\frac{a+b+c}{a} - 1 \right) \tag{13.3}$$

where a, b, c are the lengths of the 3 axes of the pseudo-ellipsoid (see section 13.3.2).

The Q_{R8} Geometric Factor

Finally, another 1-D factor named Q_{R8} is defined by:

$$Q_{R8} = \frac{\text{True Volume}}{\text{Ideal Volume}} \tag{13.4}$$

These 1-D geometric factors are studied in Section 13.4.3 in order to establish a relationship between their values and the type of the tetrahedra defined in Section 13.4.1. To do this, we need to use a "reservoir of five types of tetrahedra" described in section 13.4.2.

13.3.2 A Geometric Representation of the Size, Shape, and Orientation of a Polyhedron

Since none of these 1-D parameters is sufficient to characterise both the shape of the tetrahedron and the accuracy of the $\boldsymbol{J}$ determination, we now introduce two parameters to characterise the shape of the tetrahedron in a 2-D parameter space.

These two parameters are derived from the volumetric tensor introduced in Chapter 12 in connection with the determination of spatial grandients. It was shown in Chapter 15 that the linear barycentric and least squares methods of determining spatial gradients are equivalent; therefore the volumetric tensor must contain all the relevant geometrical information needed to determine the spatial gradient by either of these two methods. This suggests strongly that parameters which describe the volumetric tensor will be rather useful in practice. It may also be noted that the volumetric tensor, and parameters derived from it, are valid for a general polyhedron defined by four or more spacecraft.

The volumetric tensor is symmetric. A symmetric tensor describes a quadratic form which can be represented by an ellipsoid in space; this ellipsoid has three principal axes, each lying in the direction of one of the eigenvectors of the tensor, with semi-length determined by the corresponding an eigenvalue.

We recall here the definition of the tensor R, fully defined in Section 12.4 (page 315):

$$R_{jk} = \frac{1}{N}\sum_{\alpha=1}^{N}\left(r_{\alpha j} - r_{bj}\right)\left(r_{\alpha k} - r_{bk}\right) = \frac{1}{N}\sum_{\alpha=1}^{N} r_{\alpha j} r_{\alpha k} \; - \; r_{bj} r_{bk} \tag{13.5}$$

which is the component form of equation 12.23.

When N is the number of vertices (or spacecraft), $r_{\alpha j}$ is the j component of vertex α, and r_{bj} is the mean value, over all α, of $r_{\alpha j}$. If the origin of coordinates is chosen to be the mesocentre, then the tensor R can be written

$$R_{jk} = \frac{1}{N}\sum_{\alpha=1}^{N} r_{\alpha j} r_{\alpha k} \tag{13.6}$$

R is determined uniquely from the known orbital positions of the N spacecraft. It attempts to describe the size and the anisotropy of the polyhedron (see Chapter 12).

The principle axes of the pseudo-ellipsoid are given by the eigenvectors $\boldsymbol{R}^{(n)}$ of R. If we order the eigenvalues as:

$$R^{(1)} \geq R^{(2)} \geq R^{(3)} \tag{13.7}$$

their square roots represent respectively the major, middle and minor semiaxes of the pseudo-ellipsoid:

$$\begin{aligned} a &= \sqrt{R^{(1)}} \\ b &= \sqrt{R^{(2)}} \\ c &= \sqrt{R^{(3)}} \end{aligned} \tag{13.8}$$

Thus, the volumetric tensor, and the associated ellipsoid, provide a simple way to visualise those features of the global shape of a polyhedron which are significant for the determination of gradients. For instance, an ellipsoid reduced to a sphere corresponds to a regular polyhedron, an ellipsoid reduced to a plane ellipse corresponds to the spacecraft being coplanar, and an ellipsoid reduced to a line corresponds, of course, to the alignment

of the spacecraft. The significance of the non-zero eigenvalues in the case of four spacecraft is explained in Section 12.4.3. Again considering only four spacecraft, it may be noted that, even if the volumetric tensor were to be renormalised so that the spacecraft of a regular tetrahedron actually lie on (the surface of) the sphere, for an arbitrary configuration the spacecraft would generally not lie on the corresponding ellipsoid.

13.3.3 Size, Elongation, and Planarity of a Polyhedron

The discussion of the preceding section, and of Chapter 12, clearly demonstrates the importance of the eigenvalues of the volumetric tensor with respect to both the description of the polyhedron geometry and the calculation of spatial gradients.

Three parameters are needed to describe the three eigenvalues. It is useful for these parameters to be "intuitively descriptive". One parameter may be used to indicate the size of the polyhedron, and the other two, elongation and planarity, to describe its shape. Furthermore, in general (when it is anisotropic) two directions are required to define completely the orientation in space of the polyhedron. The reasoning behind this choice of parameters is as follows:

- When the polyhedron is isotropic, all three eigenvalues are equal.
- If it is stretched, a^2 becomes greater than the other two eigenvalues; if stretched (or rather, if squeezed in the two orthogonal directions) until $b = c = 0$, the spacecraft would lie on a straight line. We define the *elongation*, or prolateness, to be $E = 1 - (b/a)$. Furthermore, the eigenvector $\boldsymbol{R}^a$ defines the direction of elongation.
- On the other hand, if the isotropic polyhedron is squashed in one direction, c^2 becomes smaller than the other two eigenvalues; if squashed until $c = 0$, the spacecraft would lie in a plane. We define the *planarity*, or oblateness, to be $P = 1 - (c/b)$. Furthermore, the eigenvector $\boldsymbol{R}^c$ defines the normal (or pole) of planarity.
- In general the polyhedron is both stretched and squashed, in mutually orthogonal directions. Together, the elongation and planarity define completely (the ratios of) the eigenvalues, and thus the physically important characteristics of the shape of the polyhedron. It remains to define a parameter to describe the size; it is convenient to use the largest eigenvector, a^2, which is always non-zero, and to define the characteristic size as $L = 2\,a$.

To summarise, the physically important characteristics of the polyhedron may be described completely by:

- **characteristic size** $L = 2\,a$ (in any convenient unit of length)
- **elongation** $E = 1 - (b/a)$
- **direction of elongation** $\boldsymbol{R}^a$
- **planarity** $P = 1 - (c/b)$
- **normal of planarity** $\boldsymbol{R}^c$.

The direction of elongation and the normal of planarity are (by definition) orthogonal, and so only three angles (e.g., the three Euler angles) are needed to describe completely the orientation of the quasi-ellipsoid in three dimensions. These three angles, plus the values of L, E and P, provide a complete description of the volumetric tensor. We may note that:

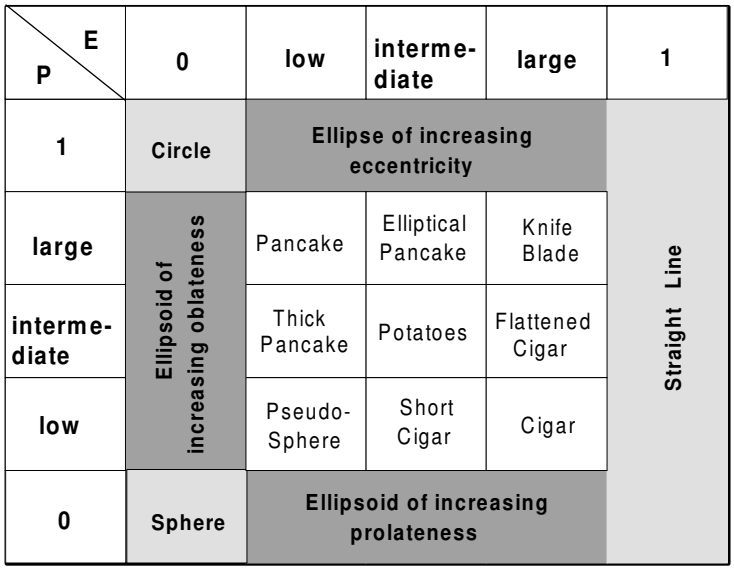

Figure 13.2: The shape of the polyhedron as a function of E and P.

- both E and P are dimensionless, and lie in the range $0 \leq E \leq 1,\ 0 \leq P \leq 1$;
- when $E = 1$, P is undefined because $b = c = 0$.

The shape of the polyhedron as E and P vary over their permitted ranges is indicated in Figure 13.2. In Section 13.4.1, we will define a limited number of general shapes to characterise the tetrahedra during the investigation of this chapter.

Note that we have defined elongation and planarity to be $1 - (b/a)$ and $1 - (c/b)$, whereas the eccentricity of an ellipse is defined by $e = \sqrt{1 - (b/a)^2}$. Now e lies in the same range $0 \leq e \leq 1$ as E and P, and the question arises as to whether elongation and planarity would have been better defined as $\sqrt{1 - (b/a)^2}$ and $\sqrt{1 - (c/b)^2}$. Then the elongation and planarity would be simply the eccentricities $e_c = \sqrt{1 - (b/a)^2}$ and $e_a = \sqrt{1 - (c/b)^2}$ of the ellipsoid respectively in the plane of planarity (containing the middle and major axes), and in the plane perpendicular to the elongation (containing its minor and middle axes). Both definitions are acceptable, but the elongation and planarity as defined above yield a more uniform distribution of points in the E-P plane. This point is, of course, entirely subjective because there is no *a priori* uniform distribution; but the statement is certainly true for typical Cluster orbits, as explained in Section 13.5.

Note that if no single parameter can reproduce all the information contained in the volumetric tensor, the converse is also true: it is not possible to express analytically the various 1-D geometric parameters in terms of the volumetric tensor, because this tensor does not describe the tetrahedron completely. A complete description would require the position of three of the apexes with respect to the fourth apex, that is, nine independent quantities of which three describe orientation and six describe shape; the symmetric volumetric tensor has only six independent quantities, of which only three describe shape.

13.4 Study of the 1-D Geometric Factors over the Tetrahedron Reservoir

We now restrict our attention again to the special case of the tetrahedron, and study several 1-D geometric factors in terms of the parameters E and P.

13.4.1 The Five Types of Tetrahedra

It is useful to limit the number of characteristic tetrahedra given in Figure 13.2 and to define only 5 representative types by means of the E and P parameters, Figure 13.3 shows where, in the E-P plane, each type of tetrahedron would be. For low values of E and P we can define a "Pseudo-Sphere-shaped geometry" (bottom left corner of the E-P diagram) corresponding to the pseudo-regular tetrahedra. For a high value of P and a low value of E (top left corner of the E-P diagram) the ellipsoid is nearly a flat circle and we can define it as "Pancake-shaped". At the opposite side (bottom right corner) we can find a long ellipsoid with a pseudo-circular section, that we can define as a "Cigar-shaped". Finally, at the top right corner, we can find tetrahedra which are both elongated and flat, and we can call this type the "Knife-Blade-shaped". Note that for elongated tetrahedron the flatness does not have much physical significance. Tetrahedra that do not belong to one of these categories or types, will be referred to "Potato type" and are located at the centre of the E-P diagram. The tetrahedra which correspond to these 5 types shown in Figure 13.3 are taken from a "five types reservoir" which is now defined.

13.4.2 Computation of a Reservoir of Five Types of Tetrahedra

Many tetrahedra corresponding to one or other of the five principal types shown in Figure 13.3 have been constructed as explained below, and placed in a "reservoir". Such a reservoir is useful in simulations in order to study the consequences of each type of configuration on the derived parameters (see Section 13.4.3). All the tetrahedra have the same mean inter-spacecraft distance:

$$\langle D \rangle = \frac{1}{6} \sum_{\alpha=1}^{6} d_\alpha \tag{13.9}$$

When computing the reservoir we start with $\langle D \rangle = 1$. The origin of coordinates of each tetrahedron is initially the mesocentre, with the axes being in accordance with Figure 13.13.

Pseudo-Sphere. There are two components of this population:

- "Regular", for which $\langle D \rangle$ is equal to each of the 6 inter-spacecraft distances d_α.
- "Random". The major part of the "Pseudo-Sphere" population is produced by perturbation of a regular tetrahedron, all three coordinates of each vertex suffering separately a random "displacement" uniformly distributed in the range $\pm\langle D \rangle \times 15\%$.

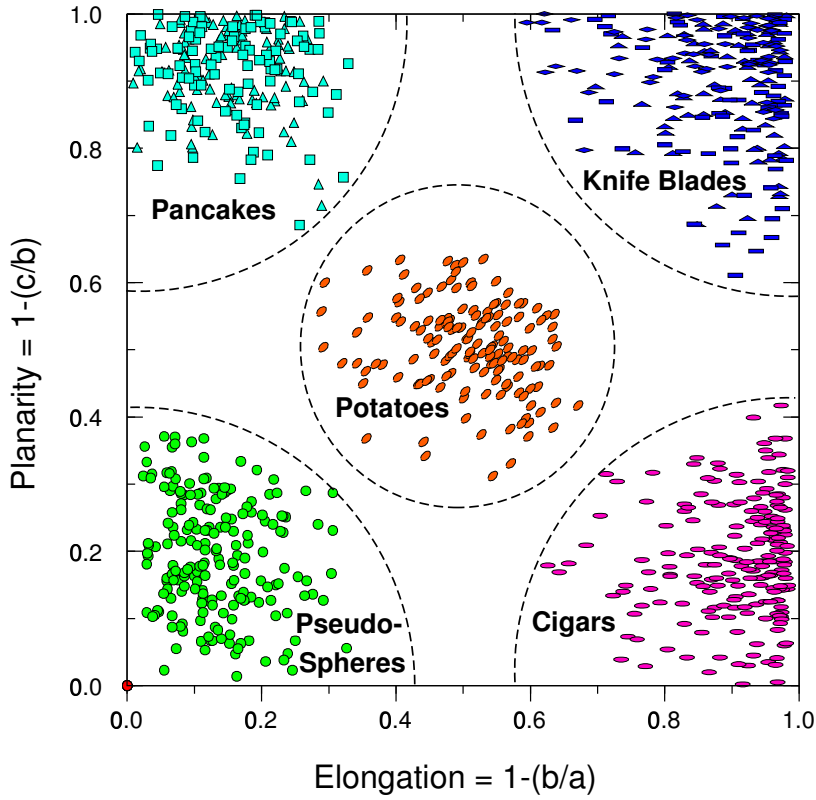

Figure 13.3: The five types of tetrahedra: Pseudo-Spheres, Pancakes, Cigars, Knife Blades, and Potatoes.

Cigar. This population is derived from the pseudo-spherical population by random elongation in the z direction, in such way as to obtain the distribution shown in Figure 13.3.

Pancake. There are two basic forms of pancake population:

- "Triangular", derived from a regular triangle in the xy plane (with the 4th vertex taken at the mesocentre of the triangle);
- "Square ", derived from a square in the xy plane.

In both cases, the three coordinates of each vertex are perturbed by a random amount uniformly distributed in the range $\pm\langle D\rangle \times 20\%$.

Knife Blade. This population contains three components:

- "Long Triangular" derived from the Triangular Pancake scaled in the x direction by a random factor in a such way to obtain the distribution shown in Figure 13.3;

- "Long Rectangular", derived from the Square Pancake scaled in the x direction by a random factor.
- a "Long Diamond", defined from a regular plane diamond in the xy plane, where the position of each vertex is perturbed by a random noise with an amplitude of $\langle D \rangle \times 15\%$ in a random direction for each cartesian component, and then scaled in the x direction by a random factor.

Potatoes. This population is derived from the Pseudo-Sphere type by elongation in both the x and z directions by different random factors in a such way to obtain the distribution shown in Figure 13.3.

After computation of the nine populations of tetrahedra defined above, the coordinates of each tetrahedron are computed with respect to its new (after perturbation of the vertices) mesocentre coordinate system. Then each tetrahedron is scaled so as to have the same mean inter-spacecraft distance $\langle D \rangle$; the value has been arbitrarily fixed at 1000 km. Finally, to randomise the spatial orientation of the tetrahedra, essential if we want to study the role of the tetrahedron direction, each tetrahedron is "shaken" in all directions, via three successive plane rotations, where the three rotation angles θ, ϕ, β are uniform random values.

To produce Figure 13.3, we have used a different number of tetrahedra for each type, as follows:

Regular	=	10
Pseudo-Sphere	=	200
Pancake (Triangular)	=	100
Pancake (Square)	=	100
Knife Blades (Long Triangle)	=	70
Knife Blades (Long Rectangle)	=	70
Knife Blades (Long Diamond)	=	70
Cigars	=	200
Potatoes	=	150

These numbers are chosen so that all five basic types contain about the same number of tetrahedra (200 for Pseudo-Sphere, 200 for Pancake, 210 for Knife Blades, 200 for Cigars, 150 for Potatoes), except the perfectly regular (10). It is worth noting that the cigar-type tetrahedra are largely over-represented; this must be taken into account in the simulations. In fact, there is no *a priori* "uniform" distribution for the shapes of the tetrahedra; any distribution which occurs in practice will be the result of a deliberate choice of orbital parameters for the spacecraft concerned.

13.4.3 The 1-D Geometric Factors and the Types of Tetrahedra

To study how the main 1-D geometric factors behave for each the five types of tetrahedra, we use the 5-types tetrahedra reservoir defined in Section 13.4.1. For each tetrahedron of the five types studied, we have computed the main geometric factors (Q_{GM}, Q_{RR}, Q_{SR}, and Q_{R8}), together with the E and P parameters. The results are given in Figure 13.4, where the x axis is the cumulative number of tetrahedra in each types. The total number of

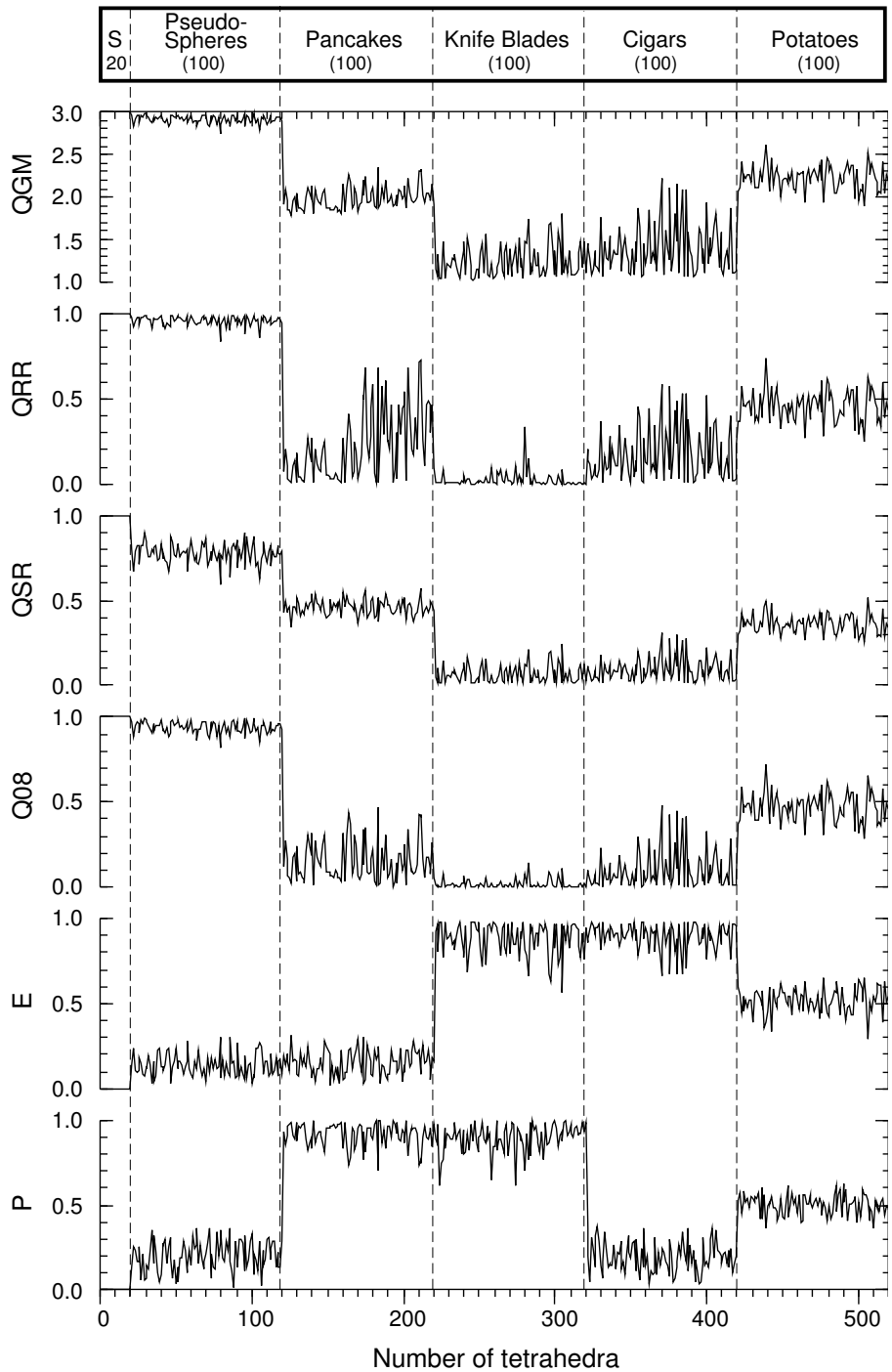

Figure 13.4: Plot of 4 1-D geometric factors and E, P parameters versus the 5 types of tetrahedron defined in Figure 13.3.

tetrahedra in each type is also indicated at the top of the figure. Note that the Q_{GM} factor varies in the (1–3) range, and the others parameters varies in the (0–1) range.

The small "S" category is the perfectly regular tetrahedron (the true sphere type). This type has been added, with a low number of tetrahedra, to verify that each 1-D parameter give the maximal value of its range (3 for Q_{GM}, 1 for the other parameters, except of course E and P).

The Pseudo-Spheres type tetrahedra gives the expected result: the Q_{GM}, Q_{RR}, and the Q_{R8} factors gives effectively a value very close to 3 for Q_{GM} and 1 for the others. Only the Q_{SR} parameter gives a value about 0.8, meaning that this kind of factor is very sensitive to the low change of a regular tetrahedron. The E and P parameters give the most real indication of the shape, in contrast to the 1-D parameters alone, and show, of course, that the elongation and planarity have low values already visible on Figure 13.3. As we will see, The E and P parameters will be obviously in accordance with the other types. Nevertheless, one should not forget that E and P alone are not sufficient to describe entirely the exact shape (see Chapter 12). They are only 2-D geometric parameters, and can indicate the main characteristics of a given tetrahedron; however, very much better that a single 1-D parameter.

The Pancake type tetrahedra gives a more complex result. The Q_{GM} factor gives the expected result, a value very close to 2, with a small variance. The Q_{SR} factor gives a value near 0.5. On the other hand, the Q_{RR} factor gives a value which varies from 0 to 0.7, the 0–0.4 part corresponding with the subtype of the triangular pancakes, and the 0.4–0.7 part corresponding to the subtype of square pancakes. The behaviour of the Q_{R8} factor is the same, but reaches only a value of 0.5. For these last two factors, it is not surprising, since these parameters are being computed from the circumscribing sphere, or for the volume of the tetrahedron, so a flat or a line tetrahedron leads to an infinite radius or a zero volume, and then a factor value near zero. Once again, one can separate the geometric factors giving information on the shape, such as Q_{GM} or Q_{SR}, from those such as Q_{RR} or Q_{R8} giving information on the accuracy of the measurement, as we will see in the next section and in Chapter 16.

The Knife Blades type gives also the broadly expected result. The Q_{RR} and the Q_{R8} geometric factors give a value very near zero, while the Q_{GM} factor, and, in a minor part, the Q_{SR} factor give a value near zero (1 for Q_{GM}) but with a rather high variance, these factors being probably more sensitive to the difference from an absolutely long and plane tetrahedron.

The Cigars type gives result which may be surprising, but can be easily understood. The Q_{GM} factor, in fact, does not make a large distinction between a cigar or a knife blade, because in the two cases the tetrahedron is long, and then give a "fractional dimension" in the range (1–2), with a high proportion close to 1. It is the same case for the Q_{SR} factor, which yields very similar result with the cigars type and the knife blade type, with a lower variance. This phenomena will be studied in details on the next section.

The "Potatoes type" gives the expected results, since the potato is an undefined shape, between the other well identified types, and gives values about 2.3 for Q_{GM} factor (0.65 if we normalise Q_{GM} in the 0–1 range) and about 0.5 for the others.

In conclusion, the study of the 1-D geometric factors with the 5-types of tetrahedra is limited, since the results are not surprising, although this kind of study allows us to be precise about the behaviour of these 1-D geometric factors with characteristic tetrahedra.

13.5 Study of the 1-D Geometric Factors with the *E-P* Parameters

The method used to study the meaning of the 1-D geometric factors is to plot the value of these parameters on a E-P diagram. To do that, we need a "homogeneous tetrahedron reservoir", whose the E and P values must cover all the E-P plane. Then, for each of these tetrahedra, the values of the Q_{GM}, Q_{RR}, Q_{SR}, and Q_{R8} geometric factors are plotted on the E-P diagram. This highlights the significance and limitations of this kind of 1-D parameter.

13.5.1 Computing an Homogeneous Tetrahedra Reservoir

The method used to compute an homogeneous tetrahedra reservoir is explained below.

Firstly, we take tetrahedra corresponding to the 9 basic forms used in Section 13.4.2. For each basic form, N_i tetrahedra are chosen in category i, we perturb (always in the mesocentre coordinate system) each vertex of the tetrahedra by a random noise with an amplitude of $\langle D \rangle \times 10\%$. Then, we define a grid of 0.1 steps in the E-P plane and decompose the E-P plane in 100 regular squares of 0.1 unit for each side. We also compute the E and P parameters for each tetrahedron, and determine the corresponding square in the E-P plane.

Secondly, we begin again this process as many time as it is necessary (with a maximum of 10 times) so that each square of the E-P plane contains about 10 tetrahedra.

Finally, to avoid a bias, all the tetrahedra have the same mean inter-spacecraft distance $\langle D \rangle$, arbitrarily fixed at 1000 km. The final result is a reservoir of about 1000 tetrahedra (ten per regular square of the 10×10 grid).

Figure 13.5 shows the result, and we can see that there is indeed an homogeneous distribution of representative points in the E-P plane. To make this reservoir, we used numbers of tetrahedra, deduced from each basic forms, as follows:

Regular	=	10
Pseudo-Spheres	=	300
Pancakes (Triangular)	=	150
Pancakes (Square)	=	150
Knife Blades (Long Triangle)	=	150
Knife Blades (Long Rectangle)	=	150
Knife Blades (Long Diamond)	=	150
Cigars	=	300
Potatoes	=	300

13.5.2 Cluster Orbit Tetrahedron in a Time Diagram

The more usual representation of the orbit of 4 spacecraft is a plot of the position of each spacecraft, and many other parameters, versus time. In Figure 13.6, we have plotted, for a typical Cluster orbit, and over one orbit, from top to bottom:

- the four geocentric distances,

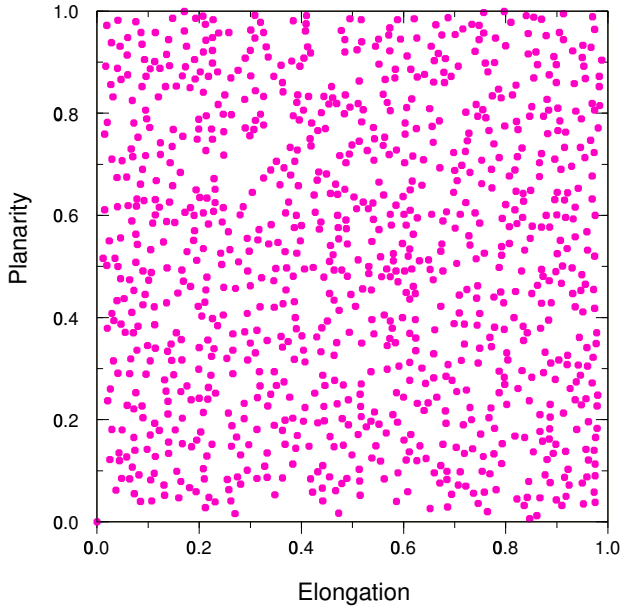

Figure 13.5: The tetrahedra reservoir used in the paper. Notice the homogeneous coverage of the E-P plane.

- the six inter-spacecraft distances,
- the volume of tetrahedron,
- the 3 geometric factors Q'_{GM}, Q_{RR}, Q_{SR} (note that Q'_{GM} factor is equal to $(Q_{GM} - 1)/2$ to have the same range (0–1) that the others parameters),
- the 3 semiaxes of the ellipsoid, a, b, c,
- the E and P parameters.

The orbit has been given by ESA [*Schœnmækers, private communication*], and was established initially on the basis of a launch in November 1995. Although the launch and the Cluster mission are delayed (failure launch of Ariane 501), the arguments remain the same. Regarding the geocentric distance, the spacecraft seem close to each other, and the four geocentric distances are superposed on the figure. One can see however that the inter-spacecraft distances vary in high proportions, and thus the shape of the tetrahedron has a strong variation along one orbit. In particular, the volume of the tetrahedron can reach a value very close to zero twice (at 26:30 UT and 33:10). This explains that, as we can see in Figure 13.6, the Q_{RR} geometric factor is very close to zero, and the Q'_{GM} and Q_{SR} reach a low value, as we have seen in preceding section. Since the minor semiaxis c of the ellipsoid is also equal to zero at these points and the middle semiaxis b has a non-zero value, the tetrahedra is fully flat (planarity parameter equal to 1, and elongation takes any value). Two other particular points can be observed, namely at 18:30 and 38:45, when $b = a$, and thus $E = 0$, corresponding to a regular sphere "flattened" in a single

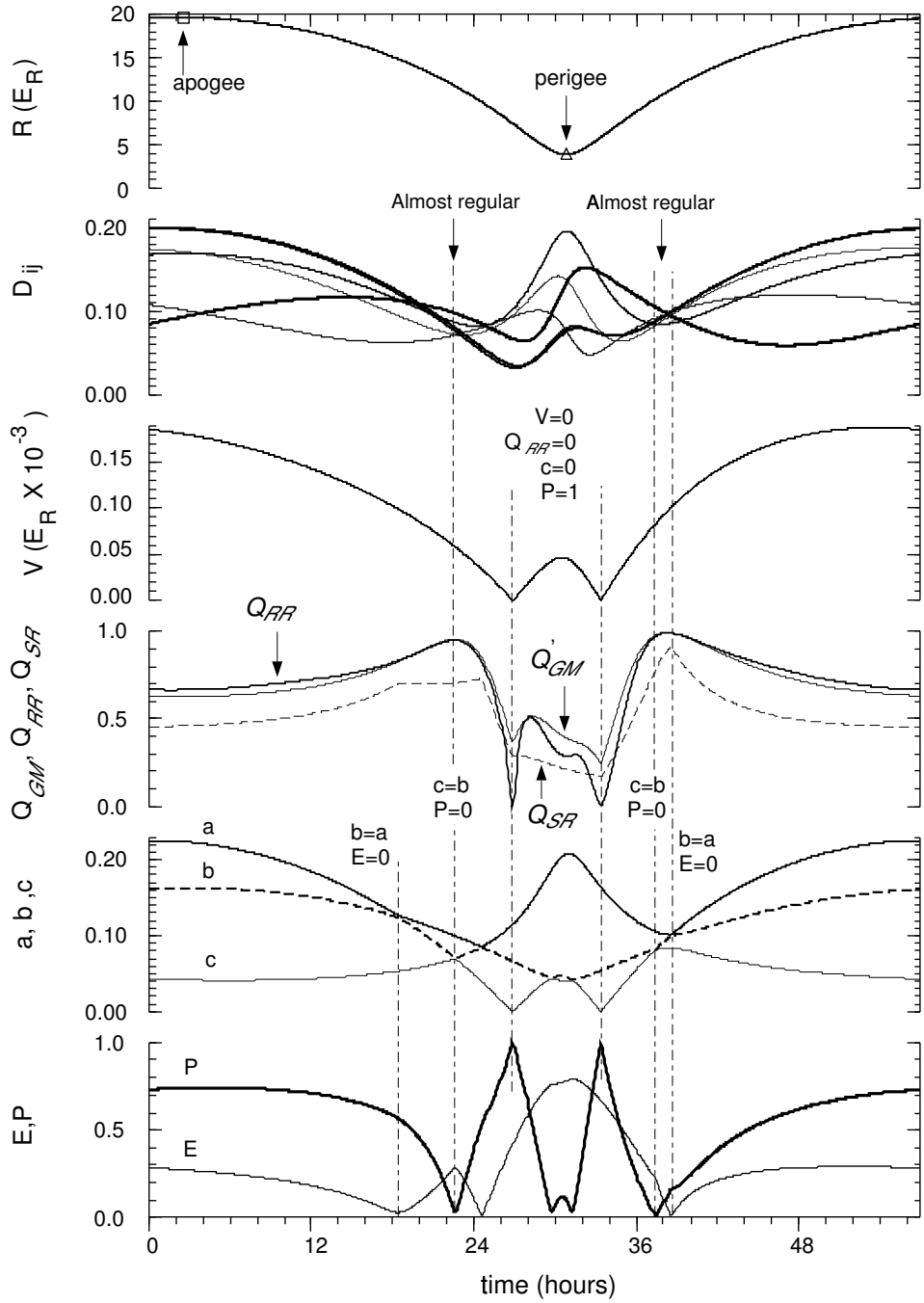

Figure 13.6: The main Cluster orbit parameters, and the 1-D and 2-D associated geometric factors, for a typical orbit of December 24, 1995 (data provided by ESA).

direction. Finally, Schœnmækers has shown that we can have two points in the orbit where the tetrahedron is regular. We find the first point at 22:30 where $P = 0$ and E have a low value. The second point is in fact a small duration in the interval [37:30–38:45] where $P = 0$ and E has a low value and then, after a short time, where P has a low value and $E = 0$. In both cases, these points can be rapidly found by examining Q'_{GM} and Q_{RR} which reach the maximum value of 1, corresponding to a regular tetrahedron. As we have seen, the Q_{SR} value has a maximum value less than 1, this parameter being very sensitive to the difference to a perfect tetrahedron.

This kind of figure can gives a good indication on the shape of the tetrahedron during the orbit of the for spacecraft.

13.5.3 Cluster Orbit Tetrahedron in the *E-P* Diagram

In order to characterise quickly the shape of the Cluster tetrahedron along one orbit, rather than plotting the tetrahedron characteristics with time as done in previous section, another way is to plot an hodogram of the successive positions of the tetrahedron in a E-P diagram. In Figures 13.7 and 13.8, the shape of the Cluster tetrahedron is computed and plotted in the E-P diagram along a whole orbit. The time step is 6 minutes, and the arrow indicates the direction of the motion. The apogee corresponds to the portion of the figure where the different points are very close together, the velocity being low and the shape slowly varying. The perigee corresponds to the portion of the figure where the points are widely spaced, because the spacecraft velocity along the average trajectory is large.

In Figure 13.7 (December 24, 1995), as we have seen in the preceding section, the tetrahedron is regular at 2 points along the orbit, the first point being located near $(E, P) = (0.28, 0.01)$, and the second point is in fact a short period, from $(E, P) = (0.21, 0.01)$ to $(E, P) = (0.01, 0.16)$. These two points where the tetrahedron is regular are, of course, located in the region of the Pseudo-Spheres type (see Figure 13.4). During the rest of the curve, the E-P parameters can take extreme values. In particular, as we have seen before, the tetrahedron is absolutely flat (P=0.99) for 2 points along the orbit, but never completely linear (the maximum value of E is 0.8 near the perigee). For another example orbit shown in Figure 13.8 (June 24, 1996) the conclusions remain the same. During the course of the Cluster mission, all possible shapes of tetrahedra are expected, and thus, simulations must take into account any possible value in the E-P plane. Thus, the homogeneous reservoir will be used for the following 2-D simulations.

13.5.4 *E-P* Diagram for 1-D Geometric Factors

The idea is the same as that in the previous section on the shape of the tetrahedron along the orbit. In Section 13.4.3, we have studied the geometric factors among the 5 types of tetrahedra defined in 13.4.1. To have a more precise idea of what the different 1-D geometric factors studied mean, we have used the homogeneous reservoir of tetrahedron defined in Section 13.5.1 to compute the values of these 1-D parameters in the E-P plane. This presentation has an important advantage: by examination of the values of these 1-D geometrical parameters in the E-P diagram, we can directly correlate the value of the 1-D geometric factors to the shape of the tetrahedron which is very best defined by the E and P parameters, although, as we have already say, E and P are themselves an approximation of the exact shape.

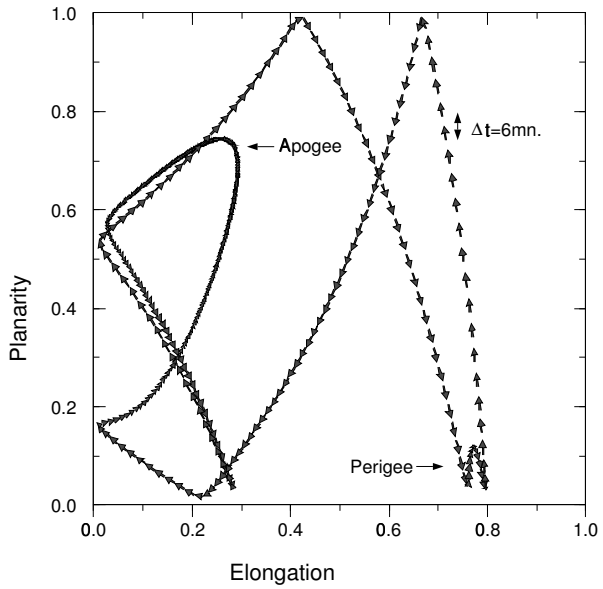

Figure 13.7: Evolution of the shape of the Cluster tetrahedron along its trajectory in a E-P diagram for December 24, 1995 (data provided by ESA).

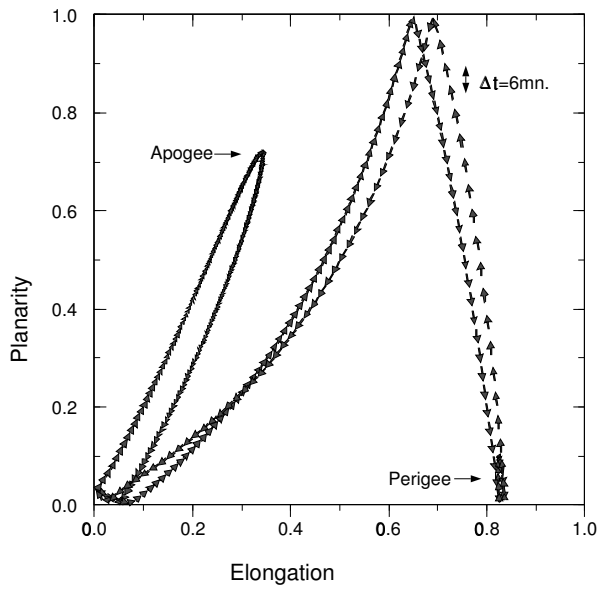

Figure 13.8: Evolution of the shape of the Cluster tetrahedron along its trajectory in a E-P diagram for June 24, 1996 (data provided by ESA).

The Q_{GM} and Q_{RR} Geometric Factors

The results are shown in Figures 13.9 and 13.10, where we have plotted in the E-P plane the values of the Q_{GM} geometric factor (Figure 13.9) and the Q_{RR} geometric factor (Figure 13.10). The size and the colour of the circles corresponds to the values of the geometric factors according to the legend given vertically on the right of each figure. At a first glance, there is an important difference between the distribution of the values of these factors in the E-P plane. Near the origin, for the low values of E and P ("Pseudo-Spheres type"), there is a similar behaviour of the two geometric factors, but, for high values of E and P, we have a difference. From the Q_{GM} factor, we can see an illustration of the fact that the P parameter becomes undefined when E is near 1 (see Section 13.3.3). In Section 13.4.3 we interpreted the meaning of Q_{GM} as the "fractional dimension" of the tetrahedron, but this kind of diagram reveals a fundamental question: does the fractional dimension exist? If we cover the sides of the E-P plane from the $(0, 0)$ origin in the clockwise direction and having in mind the Figure 13.3 describing the five types of tetrahedron, from the "Pseudo-Spheres" type to the "Pancake" type, and then to the "Knife Blades type" and the "Cigars type", the Q_{GM} value varies from 3 to 2 and then to 1 for these 4 types, corresponding to the concept of a "fractional dimension". There is no difference between a "Knife Blade" and a "Cigar", both being considered as a line shape of dimension near 1. If we consider however the transition between the "Cigars type" (D=1) and the "Pseudo-Spheres type" (D=3) to finish the clockwise tour, we reach now a fundamental problem about the "fractional dimension". These two shapes are in fact very similar since the Cigars are deduced from the Pseudo-Spheres by a strong elongation in an arbitrary direction (13.4.2), and the transition between these two shapes from dimension D=1 to dimension D=3 has to pass by the value of D=2 which, in this case, does not correspond to a plane because the planarity P is near zero. In others words, a value of Q_{GM} equal to 2 does not imply a flat tetrahedron; it could also correspond to a rather long cigar with a rounded section. On the other hand, the fact that the Q_{GM} factor does not distinguish between Knife Blade and Cigars (both being considered as a long tetrahedron) cannot be essential, because this distinction becomes impossible near $E = 1$. In conclusion, the Q_{GM} geometric factor remains a good alternative to describe, in the strong limit of a single 1-D parameter, the geometrical shape of a tetrahedron, particularly in the extreme "pancake" region, although the concept of fractional dimension must be taken with care.

Concerning the Q_{RR} geometric factor (Figure 13.10), the result is fully different for the high values of E and P. The isovalues of this factor (not plotted here, but easily guessed) are roughly decreasing with the radius $r = \sqrt{E^2 + P^2}$. This factor is not directly connected to the geometric shape of the tetrahedron, because a Pancake type tetrahedron, a Knife Blade type, and a Cigar type lead approximately to the same value for Q_{RR}. Nevertheless, this kind of parameter is rather well connected to the relative error measurement of physical parameters such as $\nabla \times \boldsymbol{B}$ for which a regular tetrahedron is often the best shape to minimise the measurement errors (at least for isotropic signature, see also Chapter 16). This property is easily explained by examining Figure 13.10. In fact, this parameter has minimum values near $E = 1$ and $P = 1$, and particularly in the region where we have simultaneously E and P close to 1 (Knife Blades). Thus the Q_{RR} factor can be seen as an expression of the *degeneration* of the tetrahedron (i.e., when E or P are close to 1), and so can be used as a real geometric factor for the physical determination of scientific

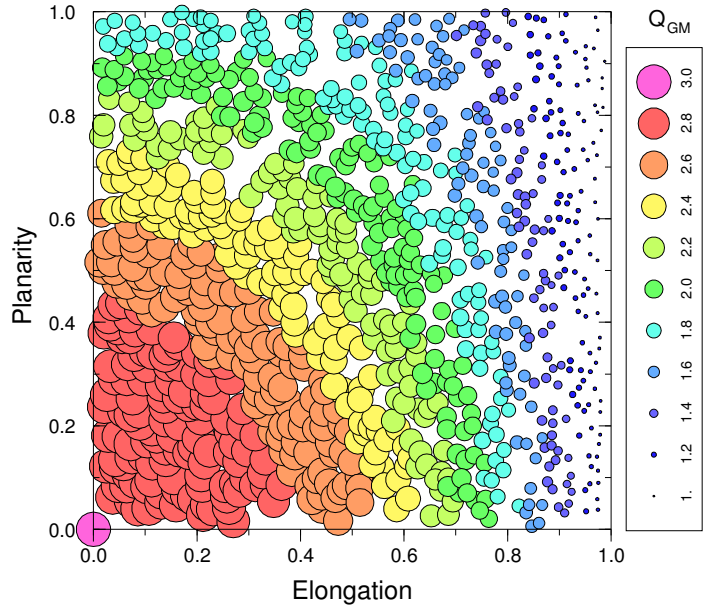

Figure 13.9: Plot in the E-P diagram of the Q_{GM} geometric factor.

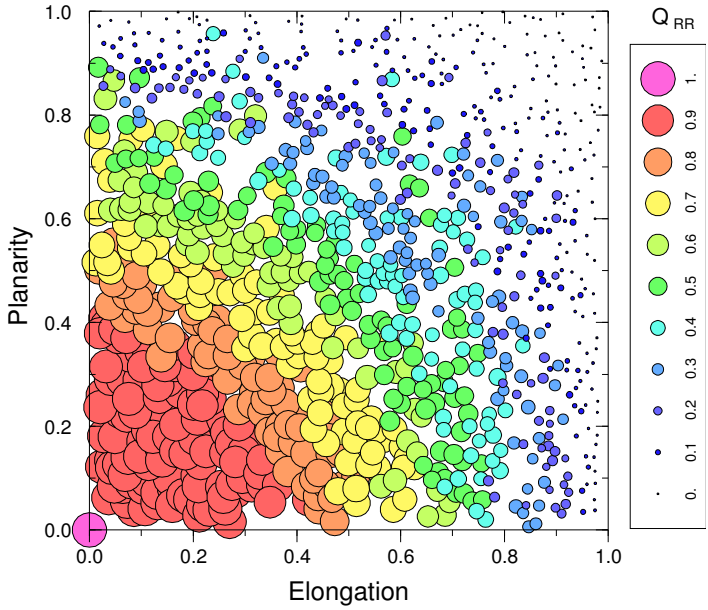

Figure 13.10: Plot in the E-P diagram of Q_{RR} geometric factor.

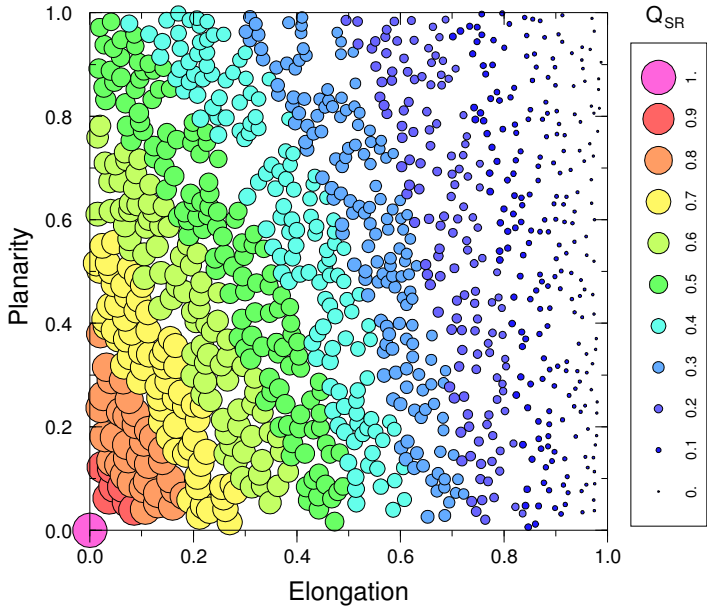

Figure 13.11: Plot in the E-P diagram of Q_{SR} geometric factor.

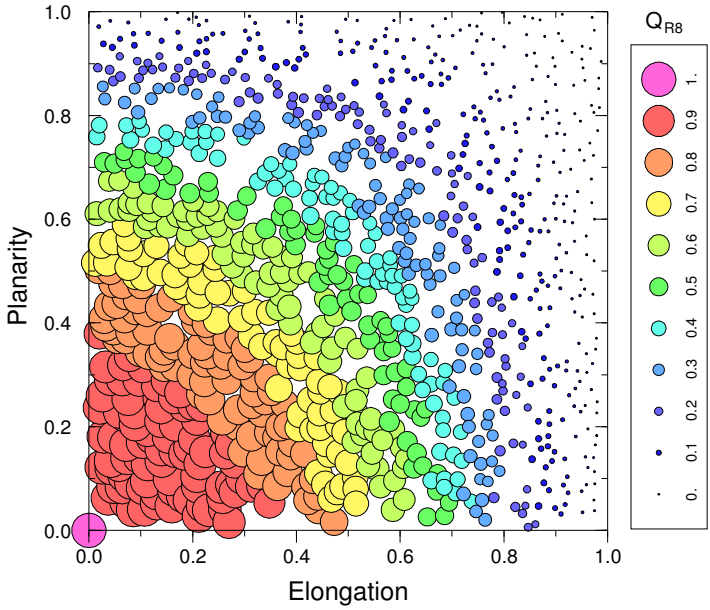

Figure 13.12: Plot in the E-P diagram of Q_{R8} geometric factor.

parameters which prefer a regular tetrahedron.

Nevertheless, for further studies, we can also define directly the degeneration of a tetrahedron as, for example, $d = \sqrt{E^2 + P^2}$, or something of the same kind.

The Q_{SR} and Q_{R8} Geometric Factors

Many 1-D geometric factors have been studied by examination of their values in the E-P diagram. In fact, we can find a lot of factors for which the E-P diagram looks the same as the two "preferred" Q_{GM} and Q_{RR} geometric factors. We present here only the Q_{SR} and the Q_{R8} geometric factors defined in Section 13.3.1 and already studied in Section 13.4.3.

The E-P diagram for the Q_{SR} factor is very similar to that of the Q_{GM} factor, as we can see in Figure 13.11. Nevertheless, there is a difference near the low values of E and P (see Section 13.4.3), where the Q_{SR} factor decrease very rapidly as soon as the E or P values are not close to zero, thus confirming the "sensitivity" of this geometric factor to a small deviation from a perfectly regular tetrahedron. Except for this difference, however, the main conclusion is the same as for the Q_{GM} geometric factor.

Concerning the Q_{R8} geometric factor (see Figure 13.12), apart from a much smoother transition, the E-P diagram of this geometric factor is very similar to the Q_{RR} one, confirming the fact that the normalised volume is a good indicator of the degree of "degeneration" of a tetrahedron.

13.6 Conclusions

The pseudo-ellipsoid, derived from the volumetric tensor of in Section 12.4.1, provides a useful and simple approach to characterise the shape of a tetrahedron, and its orientation in space. The E and P parameters allow an appropriate and easy-to-use description of this shape, and has been used to define 5 main types of tetrahedra: "Pseudo-Spheres", "Pancakes", "Knife Blades", "Cigars", and "Potatoes". These E and P parameters are used to define a 2-D geometric factor, which is a very efficient way to describe the shape and the deviation to a regular tetrahedron rather than a single 1-D geometric factor, even if it is as best as possible.

The definition of the 5 types of tetrahedron, and the making of a corresponding "reservoir of five type", has allowed us to study the response of the main 1-D geometric factors with respect to each type of tetrahedron.

On the other hand, the evolution of the shape of the tetrahedron along a typical Cluster orbit has been studied in a time diagram. By considering the different 1-D geometric factors, the length of the axes of the pseudo-ellipsoid, and the E and P parameters, we can obtain a good description of the evolution of this shape. But the introduction of the E-P diagram to plot, for instance, this orbit can give good information directly on the distortion of the tetrahedron, and its evolution. Notice that in a real case, such as the Cluster orbit, the E-P plane is well covered, and so all the values of E and P must be taken into account in any simulation.

The making of an homogeneous reservoir of tetrahedra in the E-P plane allows us to check the validity, meaning, and limits of the main 1-D geometric factors. Factors such as the Q_{GM} or Q_{SR} factors yield information on the geometrical shape, but, of course,

incompletely because of the strong limitation in single scalar values. On the other hand, the E-P diagram is not compatible with the notion of "fractional dimension" which remains an interesting concept but which has to be precisely defined. Other factors, such as the Q_{RR} factors and others, do not give real or direct information on the geometrical shape, but can be considered as the degree of degeneration of the tetrahedron and so are well related to the uncertainties in the determination of some physical parameters which prefer a regular tetrahedron.

To conclude, this study is based on the idea that a regular tetrahedron is the ideal form for good geometric measurements, but we do not forget that for special studies (for example, a boundary crossing) an alignment of the four points can be considered as the best form. Furthermore, for actual sampling of phenomena, we need to identify the relative scale and the orientation in space of the tetrahedron, which requires not only the knowledge of the length of the axes of the ellipsoid, but also their directions. When this information is unknown or is unimportant (as for an isotropic structure), since a single 1-D parameter is not sufficient to describe in a single scalar value the real shape of the tetrahedron, the use of a 2-D factor such as the E-P plane remains essential.

Appendix

13.A Calculation of Geometric Factors Q_{GM} and Q_{RR}

To calculate the geometric factors of equations 13.1 and 13.2, we need to study the geometrical properties of a tetrahedron. We consider the tetrahedron defined by four points in space numbered 1 to 4, with position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4$. Without any loss of generality, we may consider only the differences $\boldsymbol{d}_\alpha = \boldsymbol{r}_\alpha - \boldsymbol{r}_4$ in describing the points.

Area of the Sides

The area of a parallelogram bounded by two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ is given by the magnitude of their cross product; any triangle is half of a parallelogram, so its area is

$$S = \frac{1}{2} |\boldsymbol{d}_1 \times \boldsymbol{d}_2|$$

where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are the vectors for any two sides of the triangle.

We specify side α of the tetrahedron to be the one opposite vertex α: that is, it does not contain the point α.

$$S_1 = \frac{1}{2} |\boldsymbol{d}_2 \times \boldsymbol{d}_3| \quad , \qquad S_2 = \frac{1}{2} |\boldsymbol{d}_1 \times \boldsymbol{d}_3| \quad , \qquad S_3 = \frac{1}{2} |\boldsymbol{d}_1 \times \boldsymbol{d}_2|$$

$$S_4 = \frac{1}{2} |(\boldsymbol{d}_2 - \boldsymbol{d}_1) \times (\boldsymbol{d}_3 - \boldsymbol{d}_1)| = \frac{1}{2} |\boldsymbol{d}_1 \times \boldsymbol{d}_2 + \boldsymbol{d}_2 \times \boldsymbol{d}_3 + \boldsymbol{d}_3 \times \boldsymbol{d}_1| \qquad (13.10)$$

The total surface S is the sum $\sum_{\alpha=1}^{4} S_\alpha$.

Volume

The volume of a parallelepiped defined by three vectors in space is the triple product of those vectors. Any tetrahedron is 1/6 of such a figure, hence

$$V = \frac{1}{6}\,|\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 \times \boldsymbol{d}_3| = \frac{1}{6}\begin{vmatrix} d_{1x} & d_{1y} & d_{1z} \\ d_{2x} & d_{2y} & d_{2z} \\ d_{3x} & d_{3y} & d_{3z} \end{vmatrix} \tag{13.11}$$

Centre of the Circumscribing Sphere

To find the circumscribed sphere, we need the point that is equidistant from all four vertices, i.e., we want $\boldsymbol{r}$ such that

$$(\boldsymbol{r} - \boldsymbol{r}_\alpha) \cdot (\boldsymbol{r} - \boldsymbol{r}_\alpha) = r^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}_\alpha + |\boldsymbol{r}_\alpha|^2 = \rho^2; \quad \forall\, \alpha = 1, 4$$

If we take point 4 as the origin, that is, if we use the $\boldsymbol{d}_\alpha$ vectors in place of the $\boldsymbol{r}_\alpha$, then $r^2 = \rho^2$, the sphere radius, and this equation reduces to

$$2\boldsymbol{r} \cdot \boldsymbol{d}_\alpha = |\boldsymbol{d}_\alpha|^2 \quad \forall\, \alpha = 1, 3$$

This matrix equation for the centre of the sphere can be solved for the vector $\boldsymbol{r}$ and the radius of the sphere $\rho^2 = |\boldsymbol{r}|^2$. Note that the matrix $\{\boldsymbol{d}_\alpha\}$ in this equation is the same as the one whose determinant yields the volume of the tetrahedron (equation 13.11). The volume of the circumscribed sphere is then

$$V_\circ = \frac{4}{3}\pi\rho^3 \tag{13.12}$$

The Regular Tetrahedron

The regular tetrahedron of unit side is the ideal against which the true figure of the four spacecraft is to be measured. We may take (Figure 13.13)

$$\begin{aligned} \boldsymbol{d}_1 &= (1, 0, 0) \\ \boldsymbol{d}_2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \\ \boldsymbol{d}_3 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right) \\ \boldsymbol{d}_4 &= (0, 0, 0) \end{aligned}$$

Values for the regular tetrahedron of unit side length are listed in Table 13.1.

The Geometric Factors Q_{GM} and Q_{RR}

From the above quantities, it is easy to calculate Q_{GM} and Q_{RR}.

For Q_{GM}, we average the 6 distances between the 4 points to get the side L of the "ideal" regular tetrahedron, with volume $V_{\text{ideal}} = L^3\sqrt{2}/12$ and surface $S_{\text{ideal}} =$

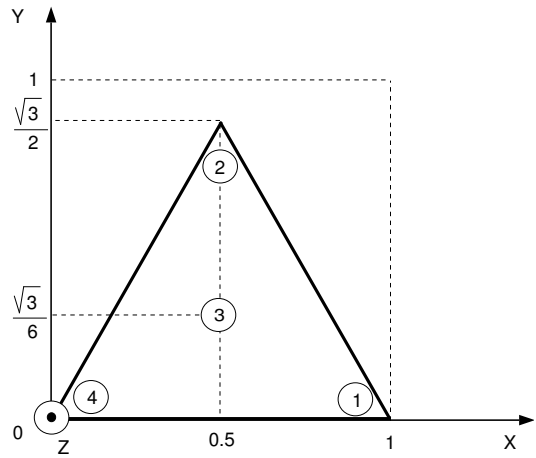

Figure 13.13: Conventions used to define a regular tetrahedron (1,2,3,4 correspond to the spacecraft position).

Table 13.1: Values for regular tetrahedron

Quantity	Value
S_α	$= \sqrt{3}/4$
S	$= \sqrt{3}$
V	$= \sqrt{2}/12$
ρ	$= \sqrt{6}/4$
$V_\circ$	$= \frac{4}{3}\pi \left(\frac{3}{8}\right)^{\frac{3}{2}}$

$L^2\sqrt{3}$. The true volume V and true surface S are found from equations 13.11 and 13.10. Then we can express Q_{GM} as:

$$Q_{GM} = \frac{V}{V_{\text{ideal}}} + \frac{S}{S_{\text{ideal}}} + 1 \tag{13.13}$$

For Q_{RR}, the radius of the circumscribing sphere is calculated from equation 13.12. The actual volume of the sphere need not be calculated, for all the factors just go into the normalisation factor $\mathcal{N}$.

$$Q_{RR} = \left(\frac{9\sqrt{3}}{8}\,V\right)^{\frac{1}{3}} \cdot \rho^{-1} \tag{13.14}$$

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Figure 13.1 has been reproduced from:

Balogh, A., Dunlop, M. W., Cowley, S. W. H., Southwood, D. J., Thomlinson, J. G., Glassmeier, K.-H., Musmann, G., Luhr, H., Buchert, S., Acuna, M., Fairfield, D. H., Slavin, J. A., Riedler, W., Schwingenschuh, K., and Kivelson, M. G., The Cluster magnetic field investigation, *Space Sci. Rev.*, **79**, 65–91, 1997.

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22, European Space Agency, Paris, France, 1990.

The elipsoid corresponding to the volumetric tensor was first defined for a tetrahedron by J. Schœnmækers of ESOC Flight Dynamics Division, who also provided the Cluster community with a code to compute the lengths and directions of the three axes of the ellipsoid defined by the four spacecraft.

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