ACCURACY OF THE ESTIMATE OF J VIA MULTIPOINT MEASUREMENTS

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ABSTRACT

A numerical modeling is carried out to estimate the accuracy of the determination of the current density via measurements of the magnetic field at four locations. Three potential sources of error are investigated: (i) the error associated with the nonhomogeneity of the current density profile over the volume defined by the 4 Cluster spacecraft. This error increases with D/R, the ratio of the intersatellite distance to the radius of the current filament under investigation. It is a deterministic error that can be partly corrected for, by appropriate methods. The ratio div B/çurl B gives an estimate of this error; (ii) the error associated with the natural noise or the noise in the electronics of the magnetometer. We have modeled this noise by random perturbations applied to the 4 x 3 magnetic waveforms. The corresponding error on $\delta J/J$ decreases as the ratio D/R increases, as long as the 4 spacecraft remain within the current filament. This error not being deterministic, cannot be corrected for, nor can it be estimated via div B/curl B. The effect of this error can, however, be minimized by an appropriate sliding average applied to the data; and (iii) the error related to the uncertainty in the determination of the distances between the spacecraft. The effect of this error is also modeled by a random process.

1. INTRODUCTION

Thanks to vector measurements made at the locations of the 4 Cluster spacecraft, it will become possible to estimate the spatial derivatives of the measured parameters, thereby giving access to new quantities. The estimate of J via curl B is probably the most often quoted illustration of this new capability, which is illustrated in the Cluster proposal (1), in the "Cluster Phase A Report" (2), and in recent papers by Balogh et al. (1988) and Dunlop et al. (1988).

Obviously, the accuracy of this estimate depends on the accuracy of the determination of various prime parameters. The purpose of the present paper is to set up a method for checking the accuracy of the estimate of the parameters of a current structure. Among the possible error sources, we will successively check against a model the effect of (i) a lack of knowledge of the intersatellite distance, (ii) the intrinsic noise of the magnetometer and/or the uncertainty in the knowledge of its positioning, and (iii) the error made when replacing spatial derivatives by finite differences and/or by assuming that the current density is homogeneous all over the volume defined by the Cluster spacecraft. These estimates of the incidence of the various error sources upon the determination of J will be based on the comparison between J_M, the model current density, and J_C, the value that Cluster would measure, given the various uncertainties. While useful for design purposes, this comparison will not be possible with real measurements. Then, it would be useful to identify an estimator of the various types of errors. We investigate potential candidates.

2. METHOD

• To make easier the interpretation of the results, we used all along the paper the same model: a cylindrical current tube, the axis of which is parallel to the Z direction defined by spacecraft 1 and 4. The current density is either constant (10^8 A.m^{-2}) up to $R = 10^3$ km or gaussian-shaped with $J_{max} = 10^{-8}$ A.m^2 and $R = 10^3$ km at J_{max}/e.

• Second, we let the four co-moving spacecraft go accross this filament and use the 4 x 3 magnetic components they would measure, to estimate J via curl B, as a function of the distance between the spacecraft. Two methods have been used to calculate J from the 12 magnetic components (i) curl B and hence J is estimated via finite differences between the measurements made at the four spacecraft locations, and (ii) J is estimated via contour integrals calculated over 3 of the 4 triangles that can be defined by relating the four spacecraft by segments of straight lines. Expressions used in each case are given in the appendix. Not surprisingly, the two methods give the same results. In the present paper, we have used the contour integral method.

• Third, we apply random perturbations to (i) the intersatellite distance, and to (ii) each of the 4 x 3 magnetic components measured by the fluxgate magnetometers. The effect of the various error sources is estimated by computing

$$\frac{\delta J}{J} = \frac{J_C - J_M}{J_M} \quad (1)$$

$$\delta\theta = (\bar{J}_C, \bar{J}_M) \quad (2)$$

where J_C is the current density "measured" by the four spacecraft and J_M the theoretical value of the current density at the center of gravity of the 4 spacecraft. δJ is the error in the modulus of J and $\delta\theta$ in the angle between J_C and J_M.

In the simple case where the current density is a step function of the radius of the current filament, the effect of an uncertainty ΔB in the estimate of each magnetic component leads to

$$\frac{\Delta J}{J} = 2\sqrt{3}\ \frac{R\ \Delta B}{D\ B_s} \qquad (3)$$

where the intersatellite distances are assumed to take a single value D. $B_s = \mu_o\ JR/2$ is the toroidal component of the magnetic field, and $\Delta J = \max(\delta J)$ is the maximum of the error in the estimate of the modulus of J. Similarly,

$$\Delta\theta = \arc\sin\left(\frac{\Delta J}{J}\right) \qquad (4)$$

is the maximum error in the angle between J_C and J_M.

3. RESULTS

3.1 Effect of the distance between the spacecraft

For a constant current density within the current tube, there is no error associated with the finite distance between the spacecraft, as long as they are all located inside the current tube. Let us now consider the more realistic case of a gaussian-shaped current density.

First we consider the case where $D_4 = D_3 = D_2$ =250 km, all distances being referred to spacecraft 1 and d_2, d_3, d_4 being the distances between spacecraft 2, 3, 4 and spacecraft 1. Figure 1a shows J_C and J_M as a function of time in the spacecraft 1 frame. Here, spacecraft 1 crosses the center of the current filament. As the distance between the spacecraft (250 km) is much smaller than the size of the filament, the finite distance between the spacecraft introduces little errors; J_C is very close to J_M.

For $D_4 = D_3 = D_2$ = 500 km, the difference between J_C and J_M now becomes significant, as evidenced in Figure 1b. Yet the estimate is still very good.

The Regatta spacecraft is foreseen to be located at a distance significantly larger than the distance between Cluster spacecraft. In Figure 2a, we investigate the effect of having one spacecraft at a larger distance, D_3 = 1500 km, than the distances between the 3 other spacecraft, $D_2 = D_4$ = 500 km. The spacecraft configuration is sketched in Figure 2b. The difference between J_C and J_M is quite large, especially close to the maximum.
In spite of the fact that one of the spacecraft (s/c 3) was crossing the current tube at a distance larger than R, the typical radius, the estimate remains quite good.

3.2 Uncertainty in the knowledge of the intersatellite distance

In order to assess the effect of the uncertainty in the knowledge of the distance between the spacecraft, we have applied perturbations to D_4, D_3 and D_2. These perturbations are extracted at random from a gaussian reservoir characterized by a root mean square $< \delta D >$. Then, in Figure 3, we plot the relative error on J versus $< \delta D > /D$. In the case of a step function for the current density profile, we get a regular increase of $\delta J/J$ as $< \delta D > /D$ increases, as shown in Figure 3a. The broken lines indicate the root mean square $< \delta J > /J$. From the figure, we get :

$$\frac{\delta J}{J} \approx 0,5\ \frac{< \delta D >}{D} \qquad (5)$$

The interspacecraft distance is $D_2 \approx D_3 \approx D_4 \approx 500$ km. Spacecraft 1 crosses the center of the current filament.

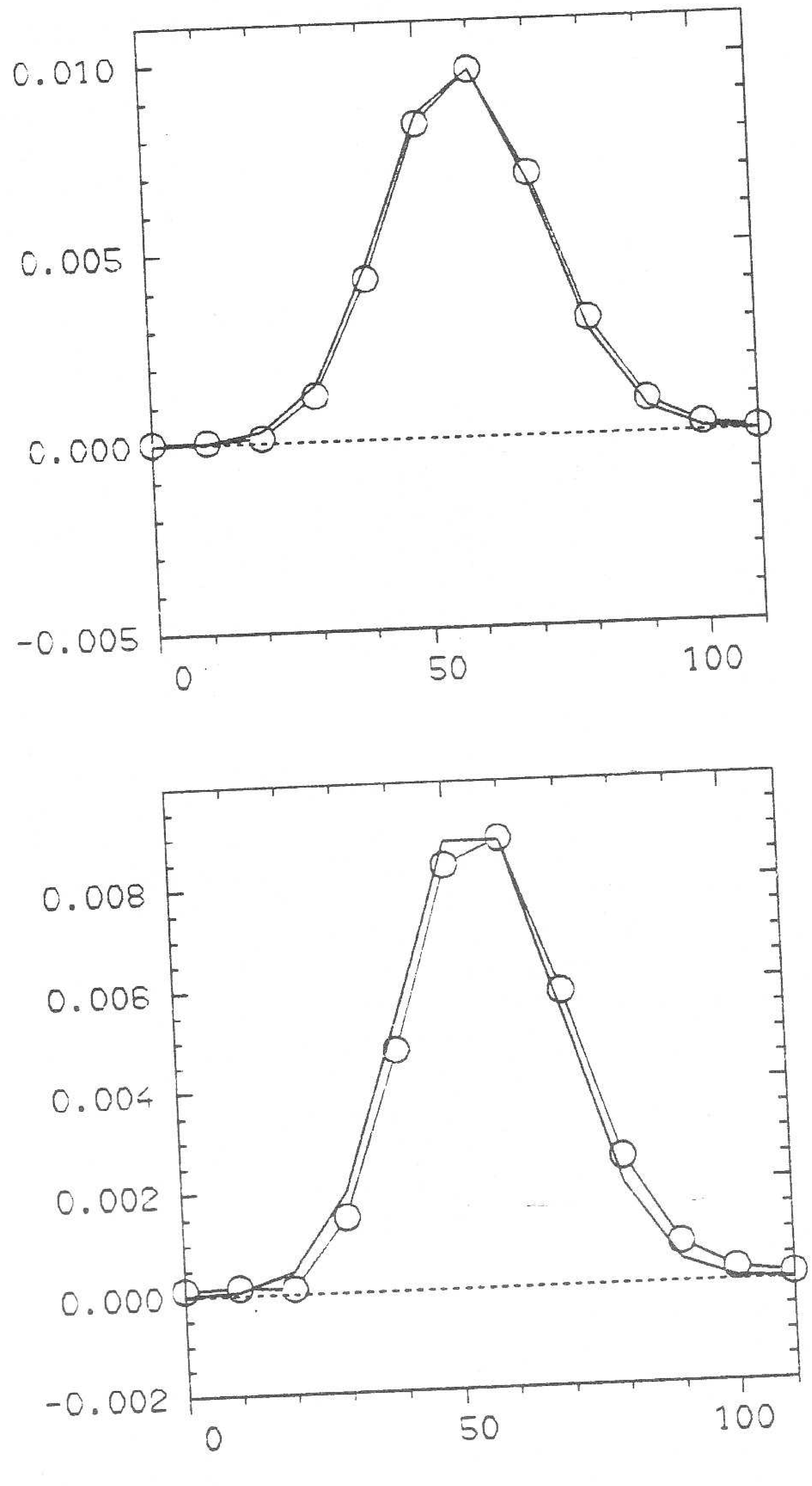

Fig. 1. Current density versus time in the spacecraft frame. Full line JM (Model), circles JC (calculated from simulated Cluster measurements). The spacecraft are assumed to be comoving at 50 km/s-1 with respect to the current filament which has a radius R = 1000 km.
a) D2 = D3 = D4 = 250 km
b) D2 = D3 = D4 = 500 km

The same geometry is used in Figure 3b but the current density now has a gaussian profile. Then the error is not null for $< \delta D >$ = 0. As discussed in the previous subsection, there is an error which is associated with the fact that the current density is not constant over the volume defined by the four spacecraft. The discreteness of the points for $< \delta D > \sim 0$ comes from the fact that the magnetic field was "measured" at 6 successive positions, as the spacecraft went across the current filament. The effect of increasing $< \delta D > /D$ is clearly seen in the figure. Up to $<\delta D> /d \sim 0.2$, the lead contribution to $< \delta J > /J$ is the error which is due to the nonhomogeneity of the current over the Cluster volume.

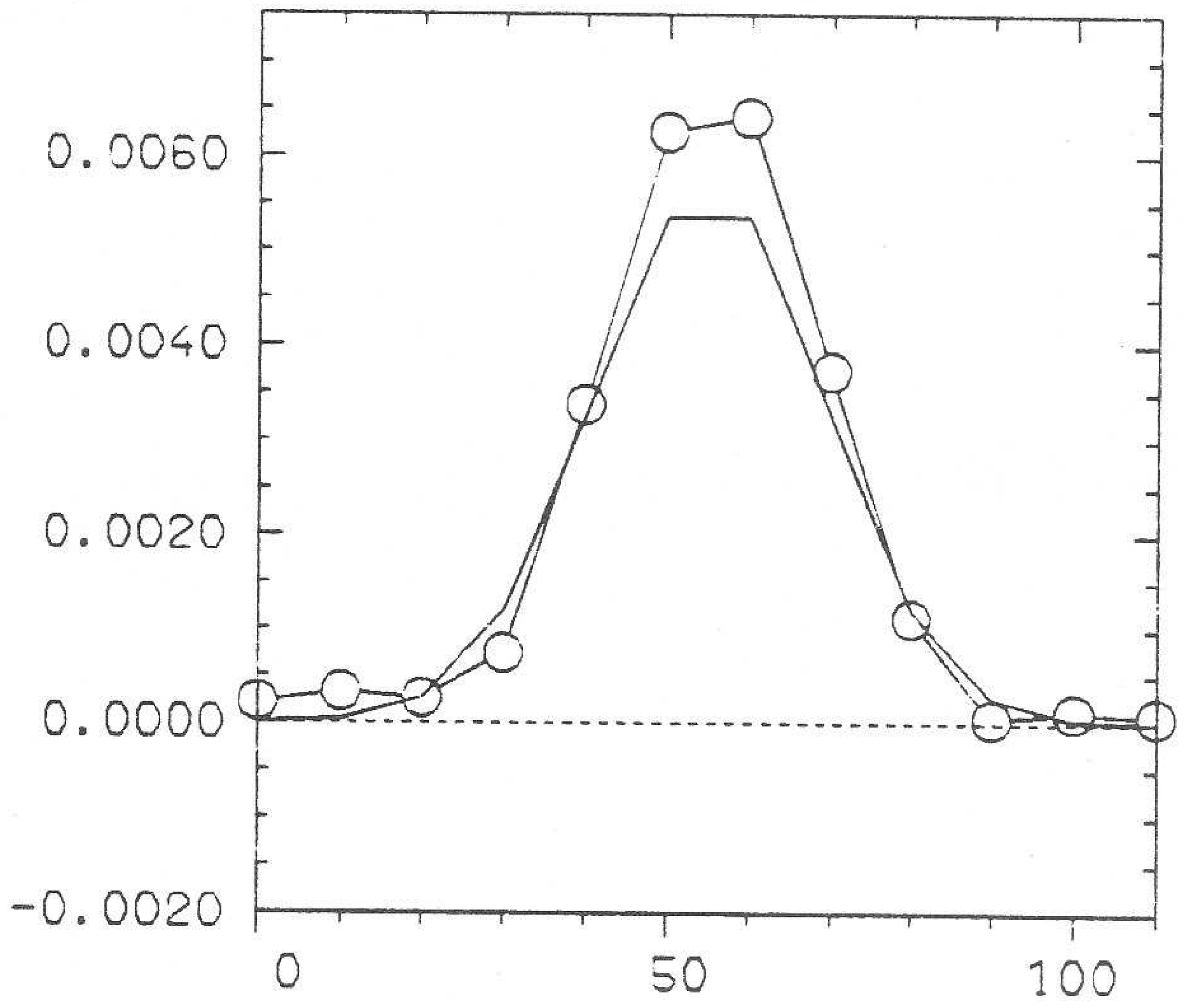

Fig. 2a *Same as 1, but $D_2 = D_4 = 500$ km, $D_3 = 1500$ km.*

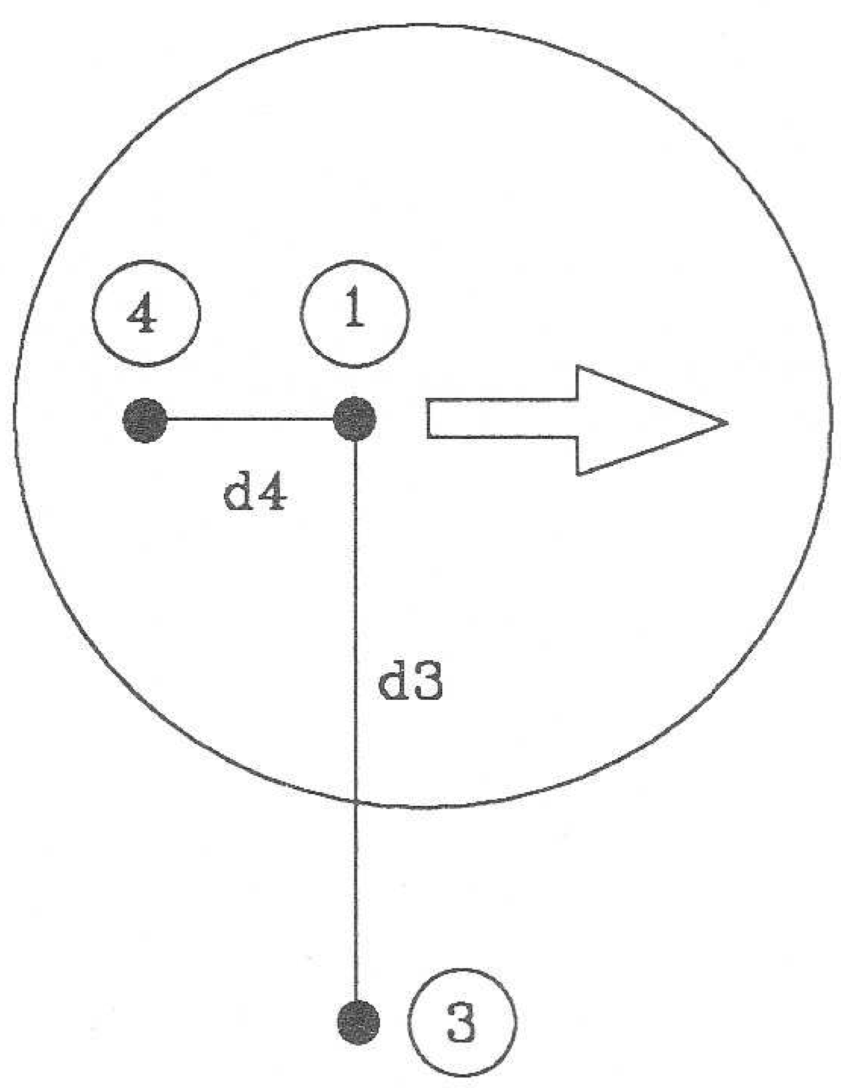

Fig.2b. Position of the satellites with respect to the current tube.

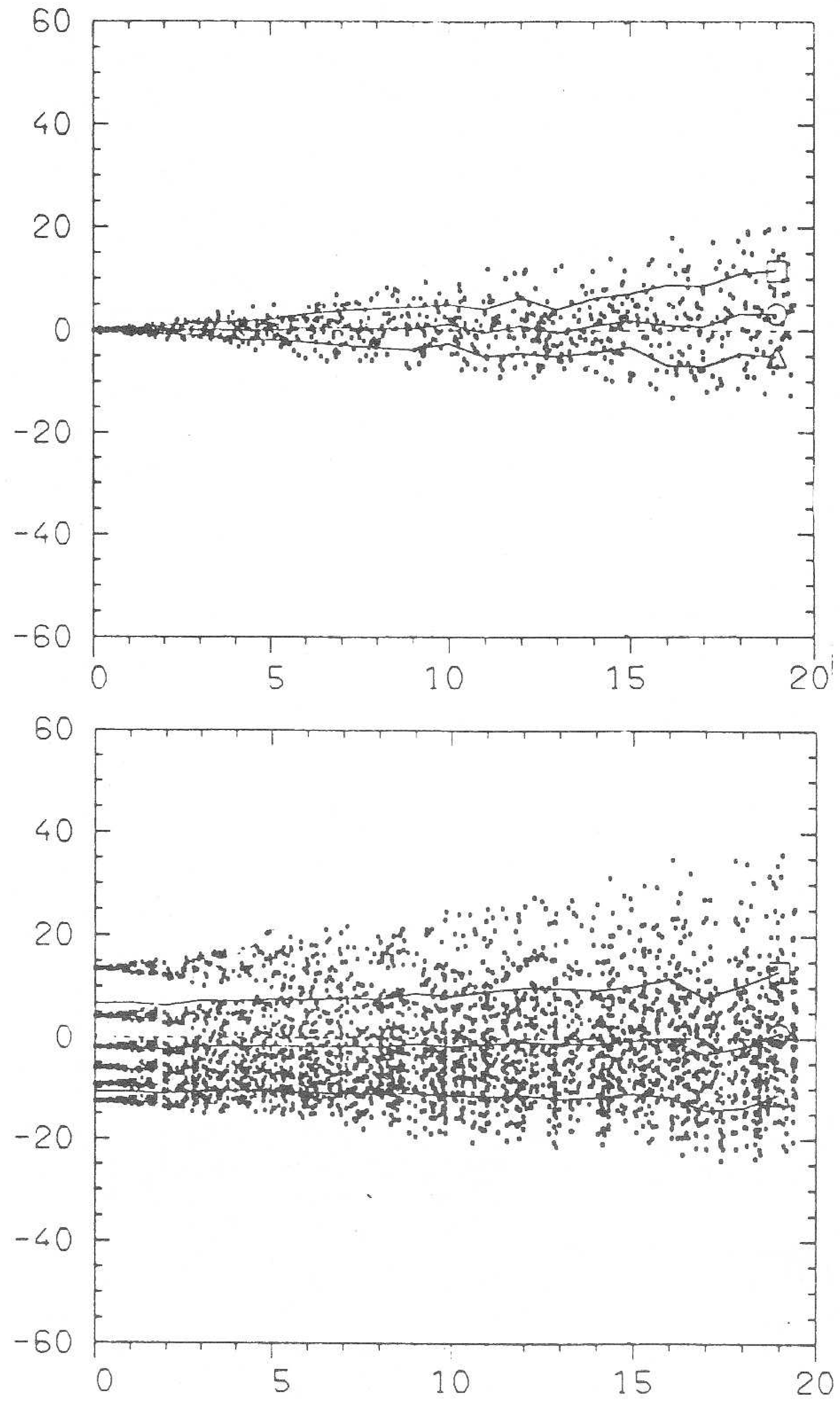

Fig. 3. Relative error $(J_C - J_M)/J_M$ (in percent) versus $<\delta D>/D$ (also in percent). Each point corresponds to a draw of one value of δD in a reservoir characterized by a given $<\delta D>$ (see text).
a) constant current density profile
b) gaussian-shaped current density profile

3.3 Accuracy of the measurements

In order to investigate the influence of a random noise superimposed on the magnetic field or the effect of the noise of the magnetometer itself, it is worth using again a statistical description. For this purpose, we have added to each of the 4 x 3 magnetic components a fluctuation δB selected at random in a reservoir with a given root mean square $< \delta B >$. Then $\delta J/J$ is plotted versus $<\delta B> / B_s$. Notice that B_s is not the total magnetic field; it is the toroidal component of the magnetic field, measured at $r = R$, typically $B_s \sim$ few nT. Figure 4a shows the result of this analysis for a constant current density. The straight lines represent the theoretical estimate given by equation 3. The broken lines correspond to the root mean square, it shows that the estimate of the current density is usually much better than what is expected from equation 3. Taking the root mean square as a realistic estimate leads to

$$\frac{<\delta J>}{J} \approx 0,5 \frac{R}{D} \frac{<\delta B>}{B_s} \qquad (6)$$

Here $B_s = 6$ nT, then if we require an accuracy of 10% in the estimate of $\delta J/J$, δB has to be less than 0,6 nT, a value which is well above the sensitivity of the Cluster fluxgate magnetometer. A more conservative requirement based on equation 3, which describes the maximum error, implies that $\Delta B \leq 0.086$ nT, which is still above the sensitivity of the fluxgate magnetometer, but is close to its absolute accuracy. Figure 4b is the same as Figure 4a but the effect of a nonuniform profile is now included. As expected from the discussion in 3.1, there is an error even for $< \delta B > = 0$. The observed quantification of the error for $< \delta B > \sim 0$ reflects the finite number of points (6) along the current profile, where the current density was computed, the best estimates corresponding to the points which are close to the maximum of the current. Clearly, the cloud of points is broader than from 4a; some of the computed points exceed the theoretical limit given by equation 3, which is not surprising since equation 3 only describes one type of error. The root mean square $< \delta J > / J$ increases slightly with $< \delta B > /B_s$. Then, as long as $<\delta B > /B_s < 0.2$, the dominant error source is the nonhomogeneity of the current between the spacecraft.

4. TEST OF THE ERROR

Up to now, we have compared J_C, the value of the current, calculated from simulated Cluster with J_M, the model current density taken at the center of gravity of the

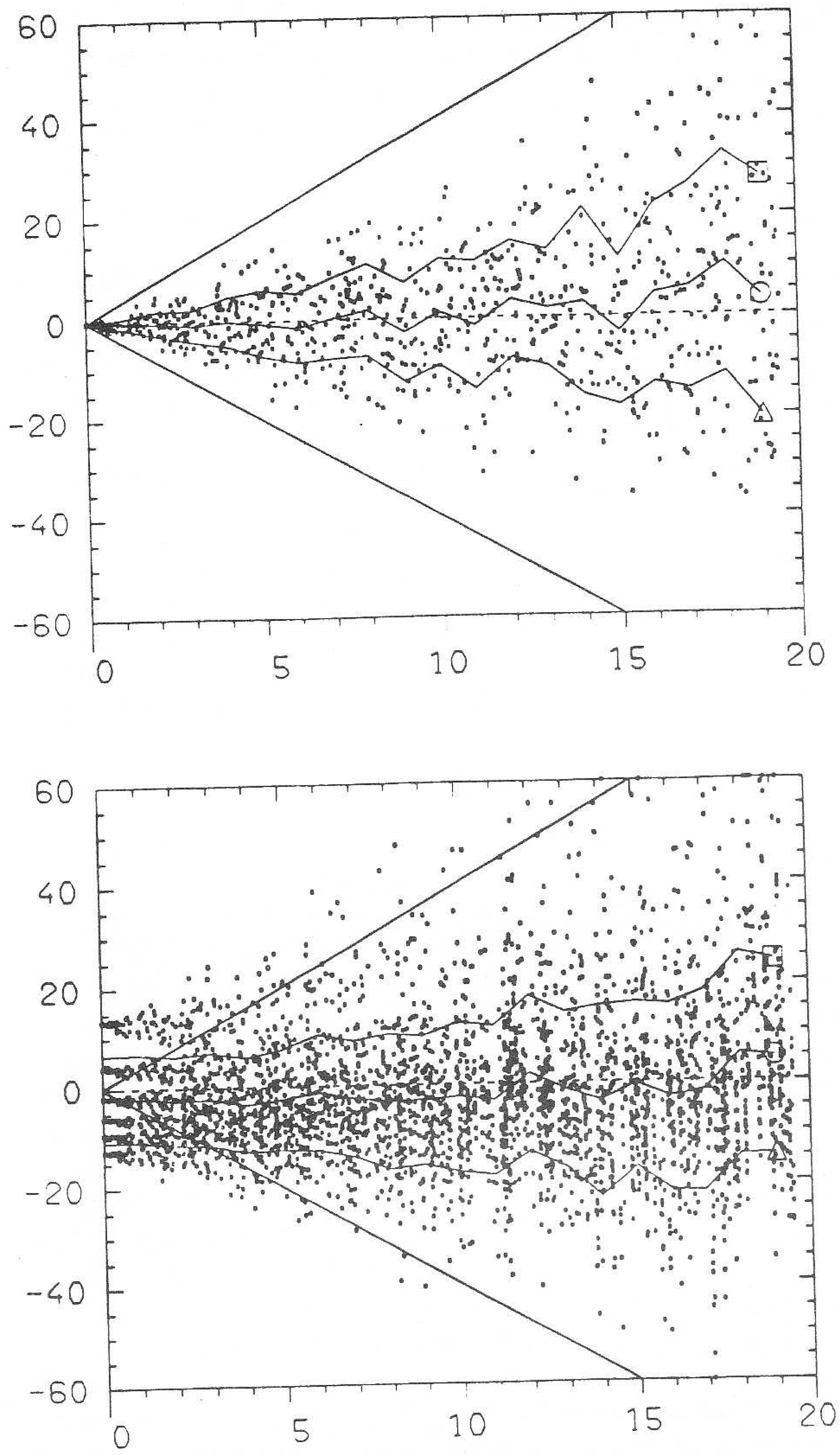

Fig. 4. $\delta J/J$ *versus* $<\delta B>/B_s$. *Each point corresponds to a draw of* δB *applied at random to each of the 12 "measured" values of B. The full line corresponds to* $\Delta J/J = 2(R/d)<\delta B>/B_s$ *(about the same as formula 3). The broken lines corresponds to the averages* $<\delta J>/J$.
a) constant current density profile
b) gaussian-shaped current density profile

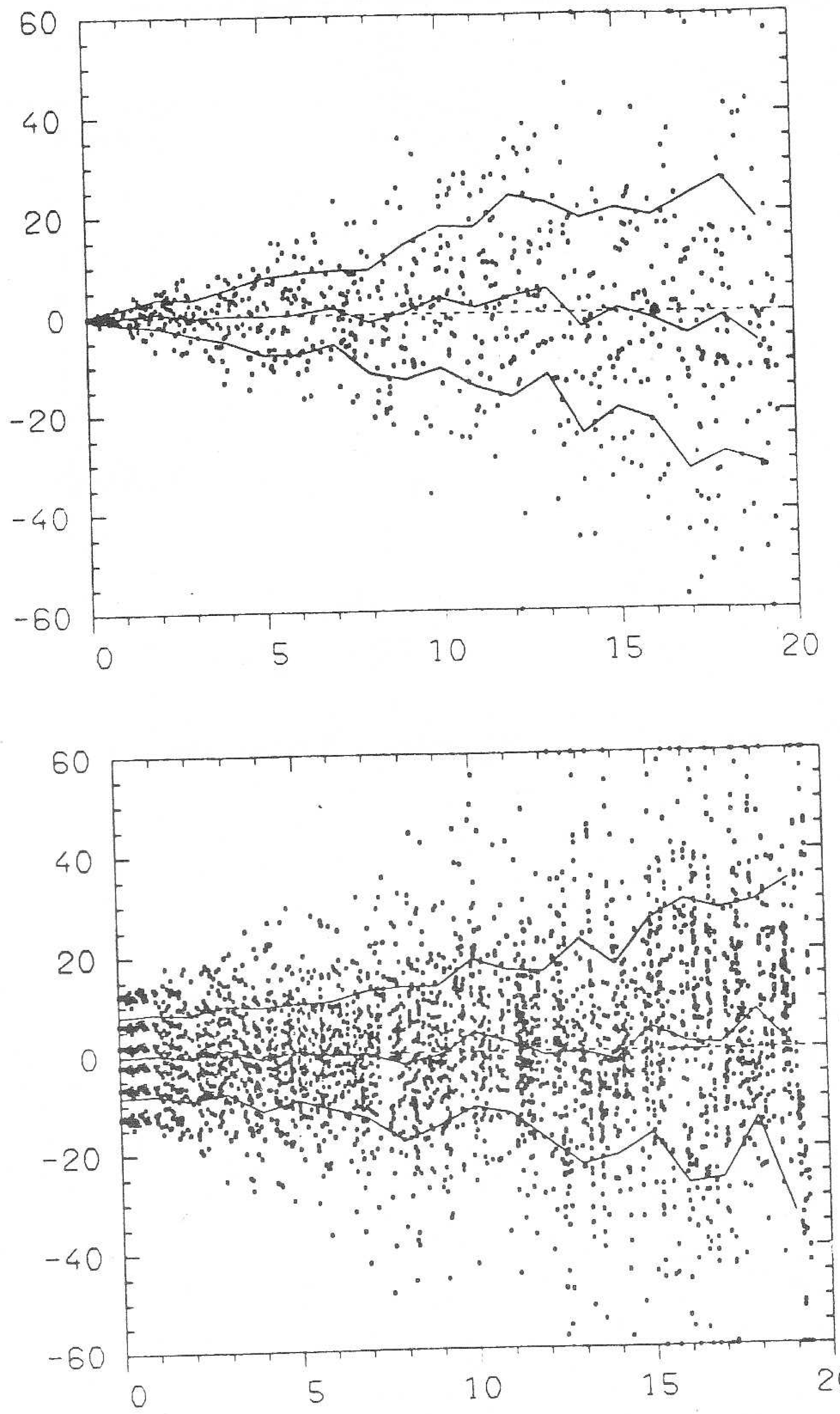

Fig. 5. div B/curl B (in percent) as a function of $<\delta b>/B_s$. *Broken line is the root mean square <div B/curl B>.*
a) constant current density profile
b) gaussian-shaped density profilemeasurements

4 spacecraft. In reality, J_M is unknown, hence it would be valuable to identify an estimator of the error. Since div B should be null for a perfect measurement, it is tempting to compute divB/curl B for estimating the error. Figure 5a shows the calculated values of div B/curl B versus $<\delta B>/B_s$, for the same parameters as in Figure 4a. The distribution of the points is about the same. In Figure 5b, the ratio div B/curl B is again plotted versus $<\delta B>/B_s$, in the case of a gaussian profile. The distribution of the points in 5b looks the same as in 4b, which suggests that, at least on the average, the ratio div B/curl B allows an estimate of the error. A case study, however, shows that the ratio div B/curl B does not provide an estimate of the error, at least when δJ is due to a random perturbation applied on prime parameters. This is illustrated in Figures 6a, b and 7a, b. In Figure 6a, J_C and J_M are plotted versus time in the spacecraft frame. The distances between the spacecraft are $D_2 = D_3 = D_4 = 500$ km; spacecraft 1 crosses the center of the current tube, the 12 magnetic waveforms are perturbed by δB drawn at random, with $<\delta B>/B_s = 0.1$. For a particular draw, Fig. 6a shows that J_C and J_M are pretty close, hence the estimate is good. The corresponding values of div B/ curl B are displayed in 6b. Even if we disregard the large values found where the current density is small, the above ratio is everywhere larger than 0.5. Hence div B/curl B is large, though the estimate of J is good.

Figure 7a shows J_C and J_M for the same parameters but for a different draw. J_C and J_M are now quite different. Yet the ratio div B/curl B is very small, as evidenced in Figure 7b. Then, in the one-by-one case, the ratio div B/ curl B does not help estimating the effect on the error on J of random perturbations on the measurement of the various magnetic components.

5. DISCUSSION

Let us now try to deduce, from the above modeling, some practical consequences for Cluster measurements.

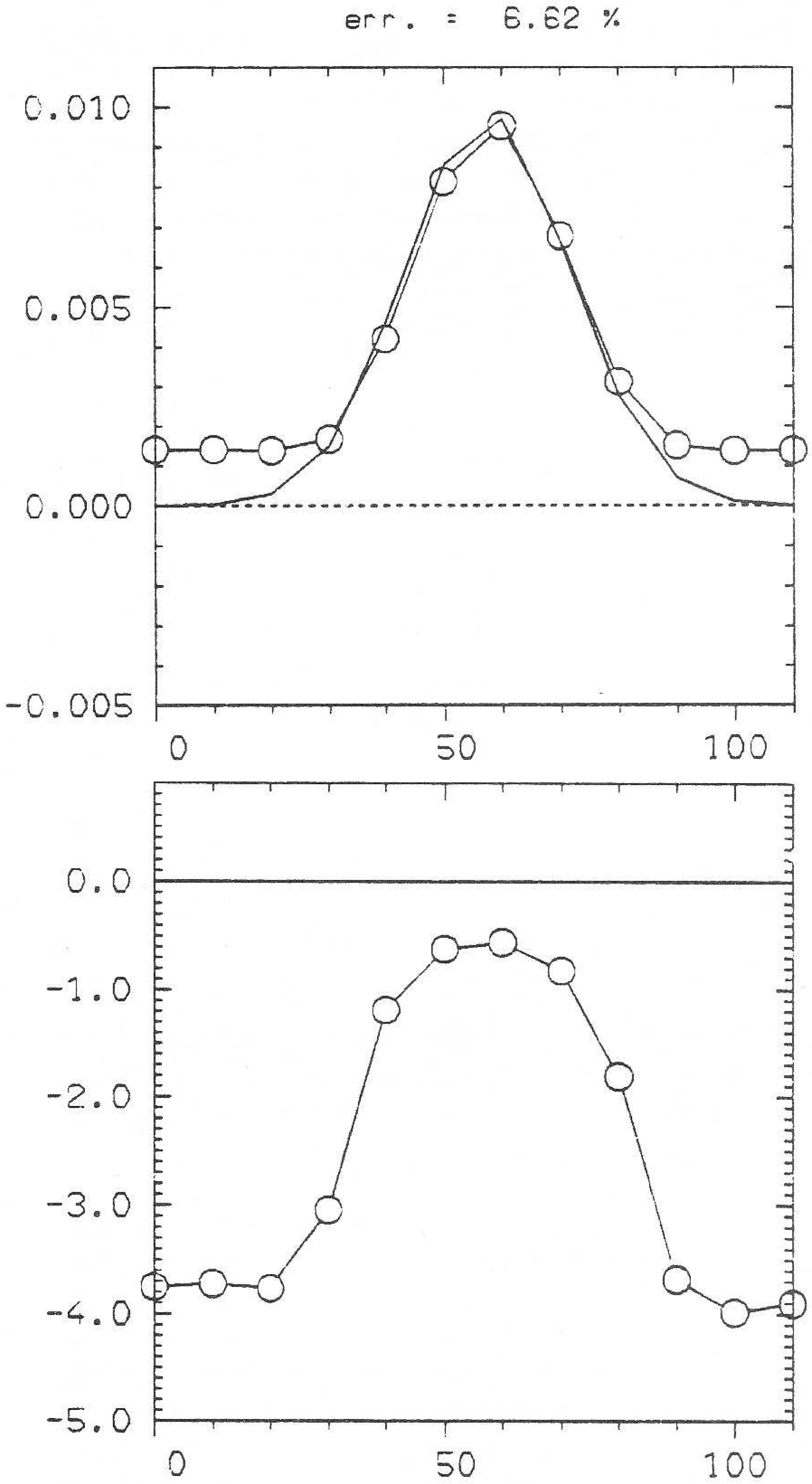

Fig 6. Test of the accuracy of the estimator
a) same as 1a, comparison between J_C and J_M (circles)
b) same as 5a (not in percent)

5.1 Requirement on the sensitivity of the fluxgate magnetometer

Formula 3 can be rewritten

$$\Delta B > 10\ \frac{4\pi \times 10^{-7}}{4\sqrt{3}}\ \frac{\Delta J}{J}\ JD \quad (7)$$

Since we want to compare this requirement with the sensitivity of the magnetometer, which is expressed in nT x $(Hz)^{-1/2}$, we have to transform ΔB from the time to the frequency domain; $\Delta B_\omega = \sqrt{\pi\Delta t/8}\ \Delta B$ for a gaussian profile with a characteristic duration Δt. Then (7) becomes

$$\Delta B_\omega > \frac{4\pi \times 10^{-7}}{4\sqrt{3}}\ \frac{\Delta J}{J}\ JD \quad (8)$$

where ΔB_ω is expressed in nT x $(Hz)^{-1/2}$. The above threshold condition is plotted in Figure 8, for various values of the product JD and for a required accuracy $\Delta J/J \leq 0.1$. In the same figure, we have plotted the sensitivity

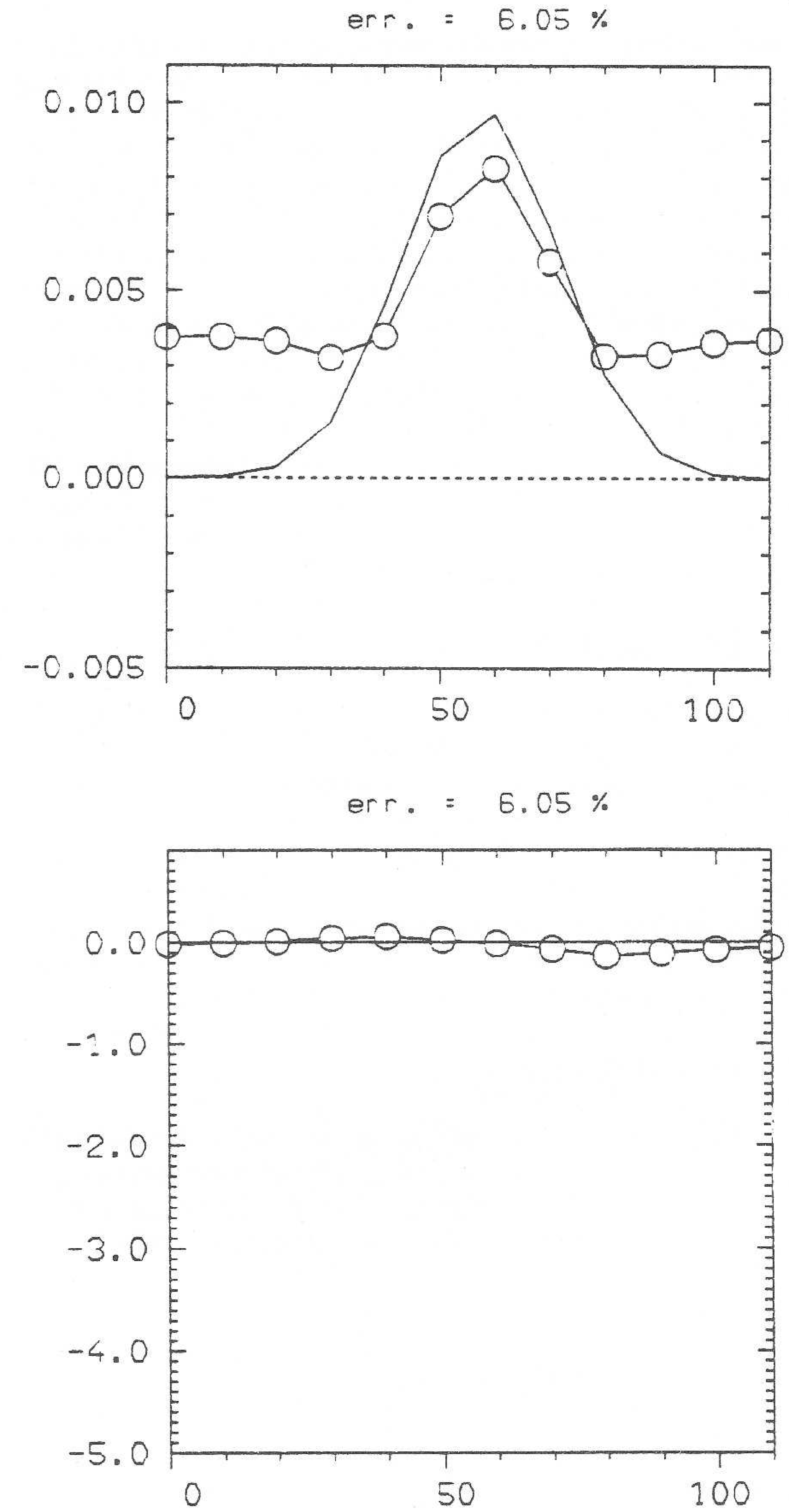

Fig. 7. Same as Fig. 6

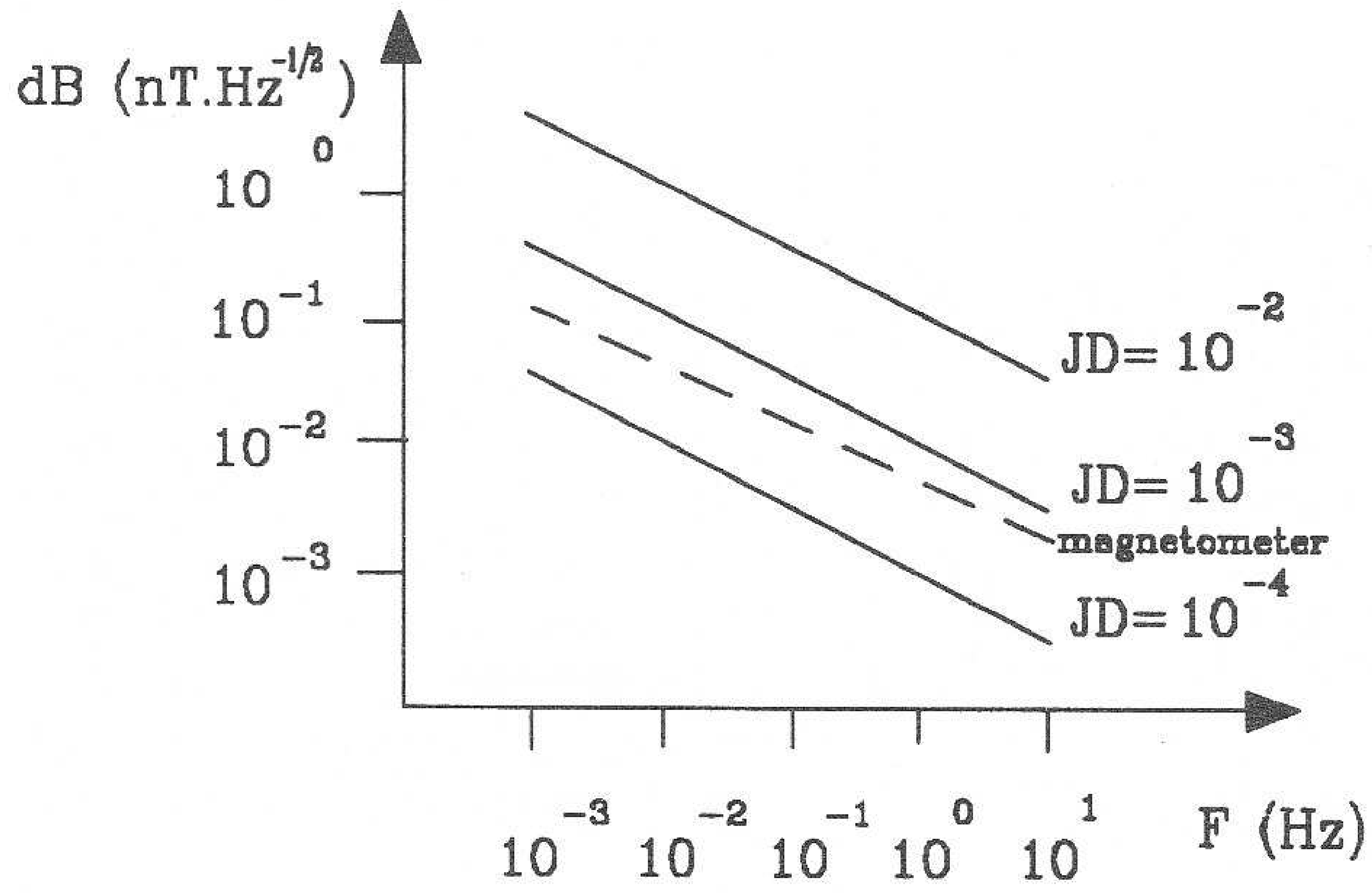

Fig. 8. Full line: threshold value for describing a current filament versus equivalent frequency $\Delta F = 1/\Delta t$, Δt being the duration of the crossing of the current filament. Dotted line: sensitivity of a fluxgate magnetometer.

of the existing Ulysses fluxgate magnetometer (Balogh et al., 1983). The fluxgates on Cluster will have better a sensitivity. From this figure, it is clear that $JD > 10^{-3}$ $A.m^{-1}$ is the limit for the detection of a current structure. Then, if $J = 10^{-8}$ $A.m^{-2}$, D must be larger than 100 km. For $D = 10^3$ km, a current $J = 10^{-9}$ $A.m^{-2}$ can be detected. It is worth noticing that the above estimate is somewhat conservative, since we have used the maximum (theoretical) error. If we use formula 6, instead, we get a velue of the threshold $< \delta B >$ about seven times larger. Then, the limit of detectability is given by $JD > 1.5 \times 10^{-4}$ $A.m^{-1}$; a current filament with $J = 10^{-9}$ $A.m^{-2}$ can be detected as soon as the intersatellite distance $D > 150$ km. In conclusion, the sensitivity of Cluster fluxgate magnetometer is sufficient to detect current filaments with a current density as low as 10^{-9} $A.m^{-2}$.

5.2 Effect of natural noise

Natural noise will be superimposed on the magnetic signature of a current filament, which could seriously restrict the applicability of the above methods and for the determination of J. It is therefore important to define a procedure for eliminating this noise. A method which is simple to implement consists in the application of a sliding average to the 12 magnetic waveforms, over a time $\Delta t = D/V$, where V is a typical velocity, determined for instance from particle measurements (see below).

5.3 Time stationarity

Assuming that a current filament is frozen into the plasma, and therefore that its perpendicular velocity is equal to that of the background, we can estimate V from particle measurements. Then the partial derivative of B, $\partial B/\partial t$, writes

$$\frac{\partial \overline{B}}{\partial t} = \frac{d\overline{B}}{dt} - \overline{V}.\nabla\overline{\overline{B}} \qquad (9)$$

where dB/dt is measured locally, at spacecraft 1 for instance, and $\mathbf{\nabla B}$ is estimated by finite differences between the measurements made at the 4 spacecraft. Then formula (9) is a possible mean of checking the time stationarity.

6. SUMMARY AND CONCLUSIONS

From the above modeling, a maximum level has been obtained for the noise of the magnetometer, in order to keep $\delta J/J < 0.1$. This noise level, which is a function of the product JD, has been compared with the sensitivity of a fluxgate magnetometer. It is concluded that, when the product $JD \geq 1.5 \times 10^{-4}$ - 10^{-3} $A.m^{-1}$ (depending upon the level of requirement), the sensitivity of the fluxgate is adequate. Then, for $D \sim 5 \times 10^5$, a value used throughout the paper, a current filament with $J \geq 3 \times 10^{-10}$ $A.m^2$ - 2×10^9 $A.m^2$ can be detected. Hence, even for low values of the current density, the sensitivity of the fluxgate is adequate to the investigation of current structures.

From the above, it is tempting to increase the distance between the satellites, to improve the accuracy of the determination of J. Increasing the ratio D/R, however, makes it essential to take into account the nonhomogeneity of the current density which results in a different kind of error, proportional to D/R. Then, two competing effects have to be taken into account for determining the optimum intersatellite distance for the investigation of a given current filament. In most cases, the error related to the nonhomogeneity of the current density over the Cluster volume will be the most stringent, which suggests that it is worth considering small distances between the Cluster spacecraft. The effect of the uncertainty in the knowledge of D has also been modeled. Not surprinsingly, $\delta J/J \leq \delta D/D$. Then, the present figure, namely $\delta D/D \leq 0.01$ or $\delta D > 10$ km gives $\delta J/J > 0.1$ as long as $D > 100$ km.

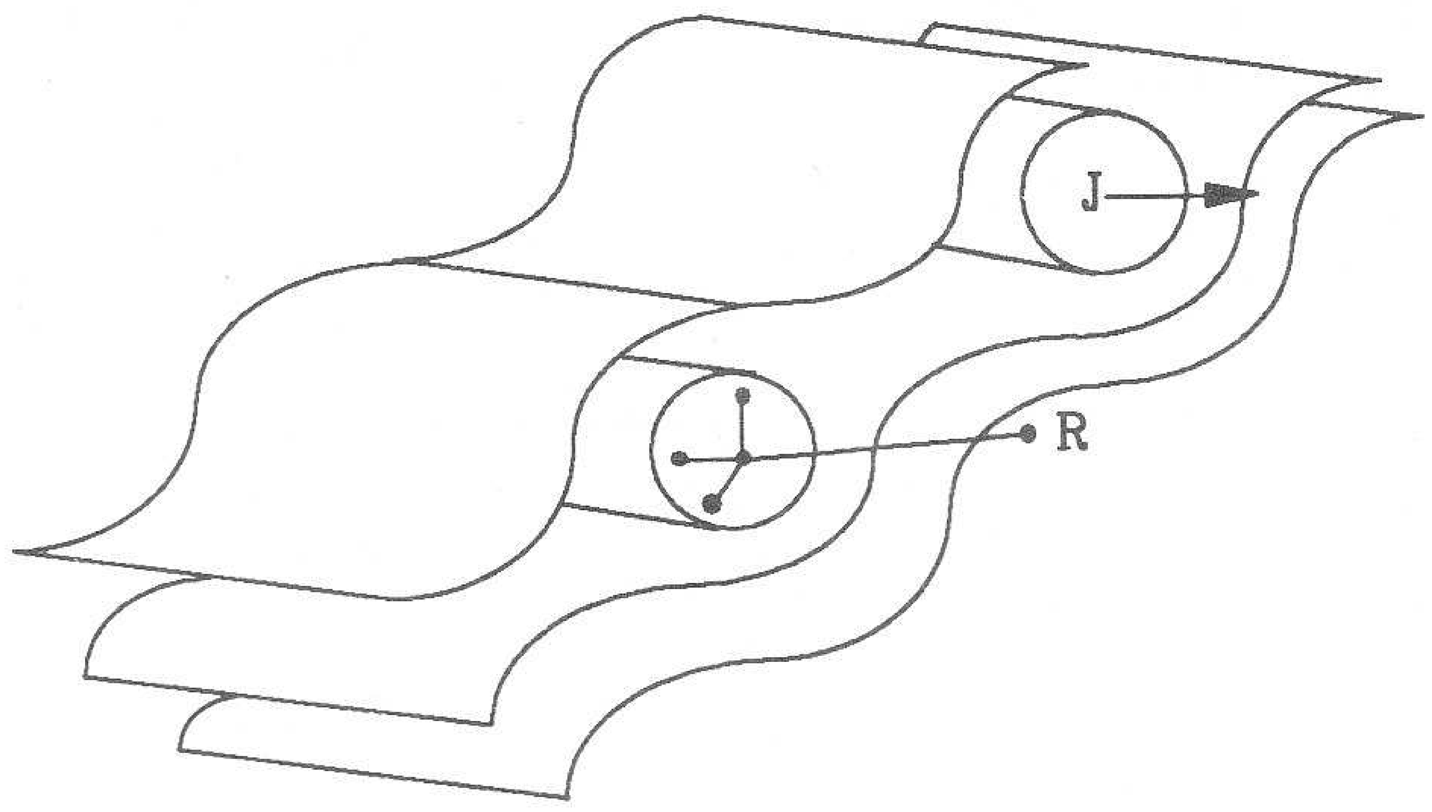

Fig. 9. Sketch showing a large-scale surface wave together with smaller scale current filaments. An ideal configuration for the four Cluster and the Regatta (R) spacecraft is shown.

The optimization of the intersatellite distance also depends upon other factors, such as, for instance, the simultaneous presence of more than one spatial scale. Figure 9 is a sketch showing a large-scale surface wave, together with the smaller-scale current filaments. The Regatta spacecraft is a unique opportunity for covering simultaneousely the large and the small scales, without loosing the specific capability of the 4 spacecraft. A ratio of 5 or 10 to 1 between these two scales seems to be a good value.

7. REFERENCES

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APPENDIX

Formula used for computing J:

1. Curl B method:

$$Je = 1/\mu_o \begin{bmatrix} (Bz3-Bz1)/d3 & - & (By4-By1)/d4 \\ (Bx4-Bx1)/d4 & - & (Bz2-Bz1)/d2 \\ (By2-By1)/d2 & - & (Bx3-Bx1)/d3 \end{bmatrix}$$

2. Contour integral method

$$Jx = (1/2\mu_o Sx) \sum_{i=x,y,z} [(B_{i3}+B_{i1})(S_{i3}-S_{i1}) + (B_{i4}+B_{i3})(S_{i4}-S_{i3}) + (B_{i1}+B_{i4})(S_{i1}-S_{i4})]$$

$$Jy = (1/2\mu_o Sy) \sum_{i=x,y,z} [(B_{i4}+B_{i1})(S_{i4}-S_{i1}) + (B_{i2}+B_{i4})(S_{i2}-S_{i4}) + (B_{i2}+B_{i1})(S_{i2}-S_{i1})]$$

$$Jz = (1/2\mu_o Sz) \sum_{i=x,y,z} [(B_{i2}+B_{i1})(S_{i2}-S_{i1}) + (B_{i3}+B_{i2})(S_{i3}-S_{i2}) + (B_{i1}+B_{i3})(S_{i1}-S_{i3})]$$

with

$Sx = d3.d4 / 2$
$Sy = d2.d4 / 2$
$Sz = d2.d3 / 2$