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original colour version of figures 2, 3, 4-a, 4-b, 5-a, 5-b in A4 size are attached

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ABSTRACT

The flux gate and search coil magnetometers embarked onboard the 4 Cluster spacecraft will be used to estimate the current density (dc and ac) within magnetospheric boundaries. The present study aims at investigating the influence of the configuration of the four spacecraft on the quality of the estimate of the current density. One expects, for instance, that a very elongated tetrahedron (one spacecraft being at a large distance from the three others) would not give a good estimate of the current density. This assumption is checked against numerical simulations involving several possible configurations (regular, almost planar, elongated, random,...). The estimate of current density is deduced from the four simulated magnetometers, with a given magnetic field model corresponding to a current density model. The uncertainty on the measurement of the distance between the spacecraft at each summit of the tetrahedron is simulated by random perturbation applied on these quantities. The current density deduced from the measured magnetic field via the Ampère law is compared with the model. Then the difference between the "measured" current density and the actual value is used to qualify the effect of the shape of the tetrahedron upon the accuracy of the measurements.The respective merits of several possible criteria for estimating the quality of the tetrahedron are discussed.

Keywords: Multipoint measurements, Curlometer

1. INTRODUCTION

A cluster of 4 spacecraft, coordinated in space, makes it possible to estimate parameters that are not accessible to single point measurements. The current density which is difficult to determine from particle measurements can be estimated from the measurements of the vector magnetic field at each summit of the tetrahedron formed by the 4 spacecraft. The estimate of the current density can be based upon finite differences between measurements of **B** at the four spacecraft locations, or upon contour integrals along the sides of the triangles formed by the 4 s/c (Balogh et al., Ref. 1; Dunlop et al., Ref. 2). Robert and Roux (Ref. 3) have shown that the two methods give identical results. The estimate of the current density **J** by these methods is only possible when the 4 spacecraft are located inside the current-carrying structure, which suggests that the distance between the spacecraft should not be too large. On the other hand, the total current crossing one face of the tetrahedron increases with the distance between the spacecraft, hence, if the current density was homogeneous in space, it would be wise to increase as much as possible the distance between Cluster spacecraft. In a realistic situation, the current-carrying structure has a certain profile, a gaussian profile for instance. Then the estimate of the current density, which relies upon the assumption that this current density is constant inside the tetrahedron, can become very inaccurate if the characteristic size of the tetrahedron exceeds the typical scale of variation of the current density. Then, as discussed by Robert and Roux (Ref. 3), the distance between the spacecraft should not be too large or too small. The optimum distance depends upon the characteristics of the current-carrying structure. Robert and Roux have defined a method based upon simulation, to estimate the accuracy of the determination of the current density. For a given profile of the current density, the (exact) value obtained analytically from the model is compared with what a cluster of 4 spacecraft would have measured, given the uncertainties in the knowledge of the positions and attitudes of the spacecraft, the noise of the magnetometers and the non-homogeneity of the current profile within the volume defined bu the 4 spacecraft. The latter error souce, which is the only one that can be tested via the calculation of div **B**, was found to be the dominant one in most cases. In the present work, we investigate, for a given mean distance betwen the 4 spacecraft, the effect of the shape (regular, almost planar...) of the tetrahedron on the accuracy of the estimate of the current density and discuss the respective merits of simple criteria that could be used to qualify the quality of the tetrahedron.

2. METHOD

The simulation method involves 5 steps described below:

- A cylindrical current tube with an arbitrary direction for the current is used. The inertial coordinate system is referred to the centre of gravity of the 4 Cluster spacecraft (Fig. 1a). The current density inside the tube is taken to be constant or gaussian. In the latter case, the maximum value is along the axis of the tube. Characteristic values are R = 5000 or 10000 km, a constant current density $J_o = 10^8$. Am^{-2}, and for a gaussian shape $J_{max} = J_o$, with a mean square deviation $\sigma = R$ (Fig. 1b). Characteristic distance between the cluster "spacecraft" (see definition below) is Dc = 1000 km.

- Once the current density model is chosen, the corresponding magnetic field vector at each point of space is calculated analytically everywhere; the corresponding values are included in the simulation program.

- The position of the 4 spacecraft inside the current tube is chosen. To get significative statistical results, a large number of configurations are studied (more than 1000). These are chosen in a "reservoir" containing the 4 kinds of configurations including (i) regular tetrahedra, (ii) "almost regular" tetrahedra obtained by perturbing the position of the summits of a regular tetrahedron by a relative distance corresponding to 30% of the distance between each spacecraft and the centre of gravity, in a random direction, (iii) an "almost planar" tetrahedron obtained via two successive deformations; first an initially regular tetrahedron is squeezed to a plane by moving one summit in the plane of the three others (at the centre of the triangle) and then the position of these four points are perturbed, as described above, and (iv) an "almost linear" tetrahedron obtained by perturbing (as described above) the positions of the 4 spacecraft initially aligned and equidistant. As shown by Robert and Roux (Ref. 3), the distance between the spacecraft has a large effect on the accuracy of the estimate of the current density; in order to avoid mixing the effect of the distance with that of the shape of the tetrahedron so as to keep the mean inter-spacecraft value $D_c = (D_{12} + D_{13} + D_{14} + D_{23} + D_{24} + D_{34})/6$ was kept constant for all selected tetrahedra. This mean distance is then referenced to as the characteristic size of the tetrahedron.

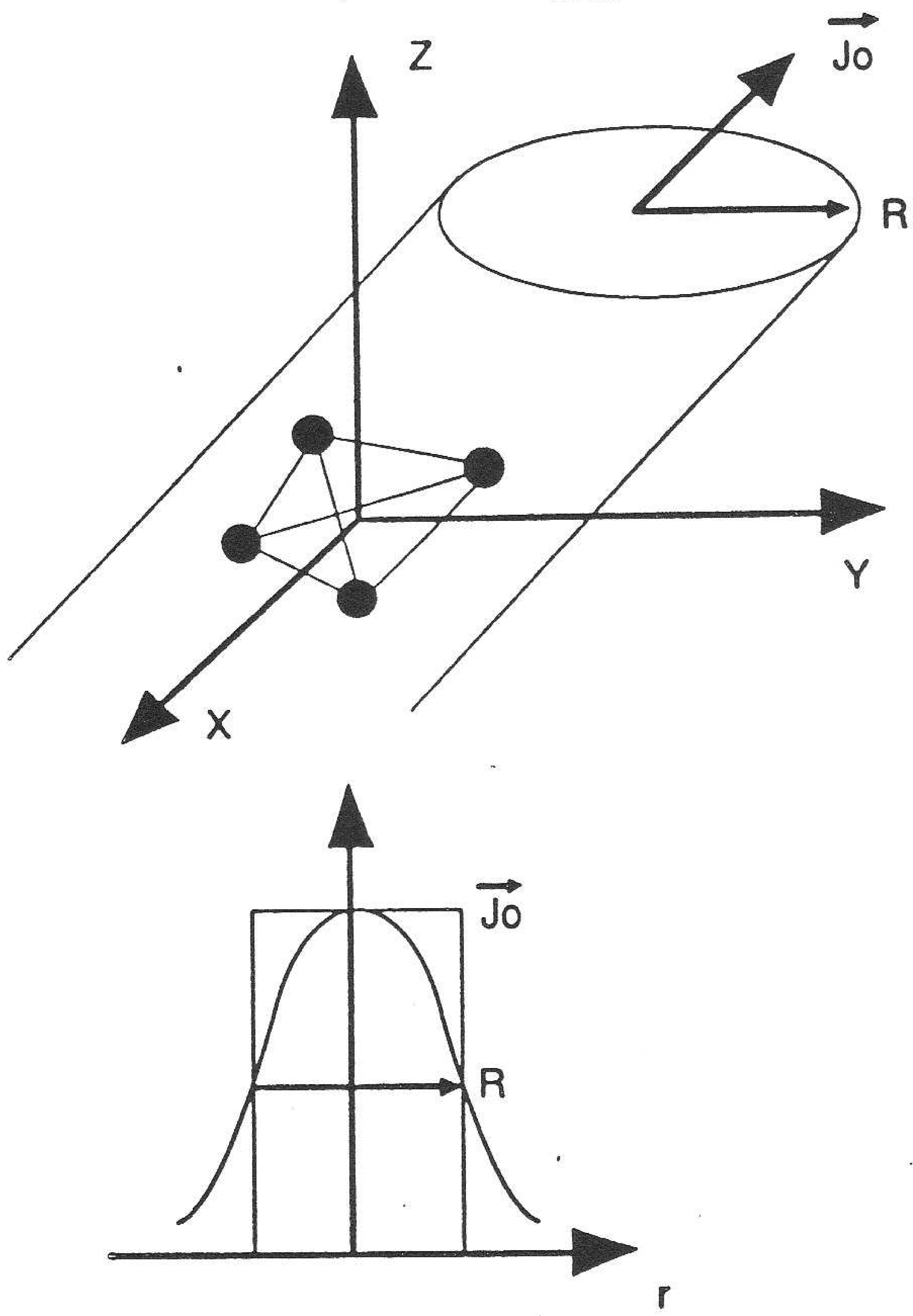

Figure 1. Top: current density model and configuration of Cluster tetrahedron: XYZ is an inertial coordinate system fastened to the centre of gravity of the 4 spacecraft, which are included inside the current structure. The direction of the current is chosen arbitrarily. Bottom: the two models used for the current density profile: homogeneous shape inside a finite cylindrical tube of radius R or gaussian shape with mean square deviation $\sigma = R$.

- Possible errors in the positions of the spacecraft are simulated by perturbing the position of each spacecraft by an arbitrary value ΔD corresponding to a given relative accuracy $\Delta D/D$, for instance 1%. D corresponds to the distance to the centre of gravity of the spacecraft. This perturbation is applied in a random direction.

- Finally, contour integrals are used to compute the current density J inside each given tetrahedron. This calculation is based upon the perturbed values of the position of the spacecraft. The error ΔJ is the difference between the current estimated from the perturbed quantities and the one given by the model: $\Delta J/J$ is the relative error. This relative error will be correlated with the uncertainty $\Delta D/D$. The dependence of $\Delta J/J$ with respect to the perturbation $\Delta D/D$ is used to qualify the influence of the shape of the tetrahedron on the accuracy of the measurement of the current density.

3. RESULTS

3.1. Uncertainty in the distance between the spacecraft

First, a constant current density model inside a cylindrical tube is used. The axis of the tube is arbitrary but kept constant during the run. As explained above, a large number of tetrahedra are drawn out of a reservoir containing the 4 families of tetrahedra; in each case, the relative error $\Delta J/J$ is computed as a function of $\Delta D/D$. In Figure 2, the relative error $\Delta J/J$ has been plotted versus $\Delta D/D$. The families of tetrahedra are indicated by different symbols, as indicated in the legend of Figure 2; the two broken lines correspond to the average values of positive and negative values of $\Delta J/J$. Not surprinsingly, $\Delta J/J$ increases regularly with $\Delta D/D$, with an average value given by $\Delta J/J \sim \pm 0.5\ \Delta D/D$. The influence of the shape of the tetrahedron is immediately clear: the points outside of the broken line (defining the average values) correspond essentially to pseudo-planar and pseudo-linear tetrahedra. The regular tetrahedra are more likely to be located inside the zone defined by the two average lines. Nevertheless, an important fraction of the non-regular tetrahedra can also give good results.

In Figure 2b; the angle between the estimated direction of J and the real value given by the model is plotted versus $\Delta D/D$. The mean value of θ (in degrees) varies approximately as $0.5\ \Delta D/D$. Concerning the shape of the tetrahedron, conclusions are the same as for $\Delta J/J$. Notice that the current value of $\Delta D/D$, guaranteed by the Cluster project, is 1% as long as the inter-satellite distance is not below 1000 km, then the values of $\Delta J/J$ will not be as large as suggested by Figure 2a.

3.2. Non-homogeneity of the current density profile

The effect of the inhomogeneity of the current density profile on the estimate of J has been investigated in the case of a gaussian-shaped current filament. The result of a simulation carried out in the same conditions as for Figure 2, but with a gaussian-shape current density profile, is shown in Figure 3. The degree of non-homogeneity can be estimated from the ratio between the characteristic size of the tetrahedron (defined in section 2) and the mean square deviation of the gaussian. In Figure 3, this ratio (Dc/σ) is set to be $Dc/\sigma = 0.5$, corresponding to $\sigma = 2000$ km.

The uncertainty $\Delta J/J$ associated with the non-homogeneity of the current profile only weakly depends upon $\Delta D/D$, as can be seen from Figure 3. Conversely, when $\Delta D/D$ is large, typically larger than 5%, the error associated with the lack of knowledge on spacecraft positioning dominates. Surprisingly, the cloud of points displayed in Figure 3 is not symmetric around $\Delta J/J = 0$: there is an upward shift. This shift is not due to a systematic overestimate of J, it is simply due to oversimplified calculations of the current from the model. Indeed, to simplify the calculation, we have assumed that $J = (\Sigma J_i)/4$, where J_i ($i = 1$ to 4) is the model current density at the level of each spacecraft. The averaged current density $\int Jdv/v$, where v is the volume defined by the 4 spacecraft should be used instead. We have verified that the sign and the amplitude of the systematic shift described above is consistent with this explanation. Hence, this shift is not associated with an uncertainty in the estimate of $\Delta J/J$ and will therefore be disregarded in the rest of the discussion.

The effect of the inhomogeneity of the current density profile is twofold: (i) there is an uncertainty even for $\Delta D/D = 0$, as described above, and (i) the fraction of the almost linear tetrahedron that leads to large $\Delta J/J$ drastically increases with the inhomogeneity parameter Dc/σ.

3.3. Quality of the tetrahedron

The accuracy of the estimate of physical parameters from 4-points measurements is a critical issue in the context of the Cluster mission. Thus, in order to help identifying the best mission scenario, it would be useful to define a simple geometric criterion that would allow an estimate of how accurate the calculation of J is, for a given spacecraft configuration. Several possible criteria have been tested by using the simulations

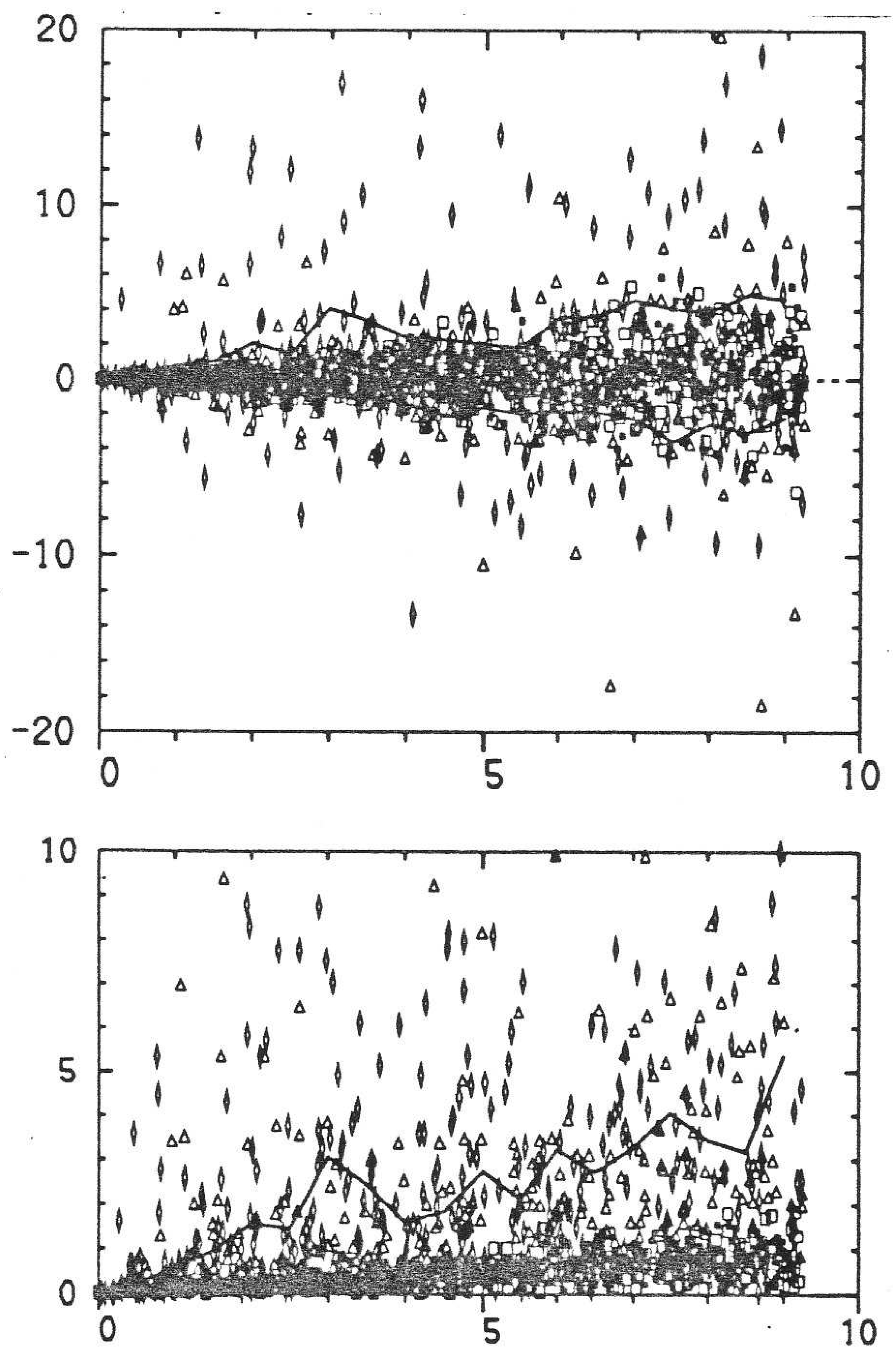

Fig. 2. Influence of the accuracy of the distance measurement on the estimation of the current density for an homogeneous current density model. Top: relative error (%) $\Delta J/J = (J_{estimated} - J_{model})/J_{model}$ versus $\Delta D/D$ (%), Bottom: angle θ (degrees) between $J_{estimated}$ and J_{model} versus $\Delta D/D$ (%) (▪ regular tetrahedra, □ pseudo-regular, Δ plane, ◊ linear)

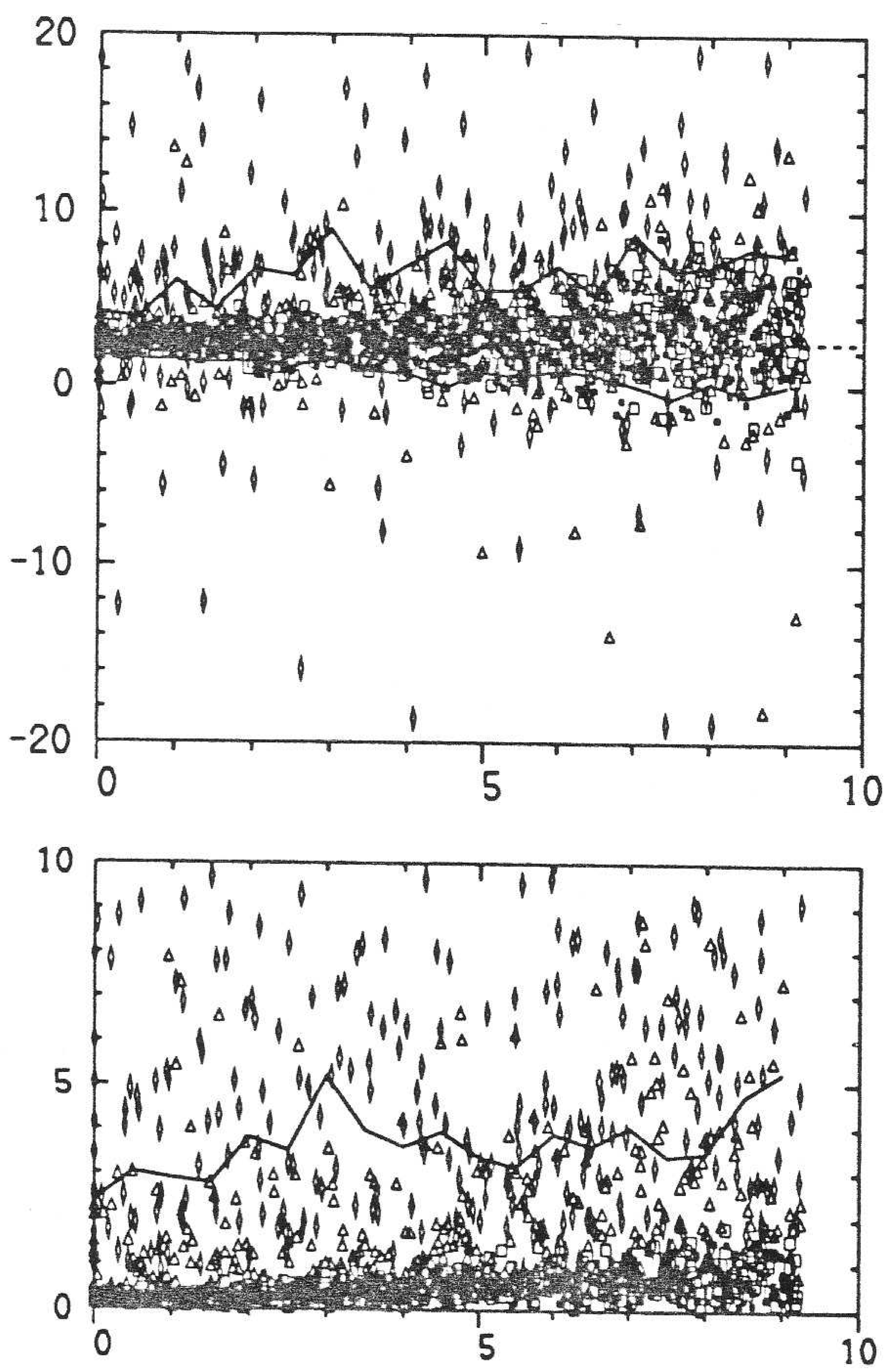

Fig. 3. Same as Fig. 2 but with a gaussian-shaped current density model with $Dc/\sigma = 0{,}5$ ($\sigma = 2000$ km)

described above. The quality parameters are chosen to lie between 0 and 1; in all cases, 1 corresponds to a regular tetrahedron but a null value can have different meanings, as described in Table 1.

Criterion	Quality parameter	Meaning of quality parameter ~ 0
A	Dmin/Dmax	two closeby s/c
B	Dmin/Dmean	"
C	Smin/Smax	Elongated tetrahedr.
D	Smin/Smean	"
E	S/Sreg	"
F	(V/Vreg+S/Sreg+1)/3	Elongated with 2 closeby spacecraft
G	(V/Vreg+S/Sreg+1)/3	"
H	Amin/Amax	planar tetrahedron
I	Amin/Amean	"
J	V/Vreg	planar or linear tetr.
K	$(V/Vsph)^{1/3}$	"
L	V/S	"

Table 1.

List of quality parameters associated with various criteria (left) and description of geometric configurations that correspond to a quality parameter equal to zero for each criterion

Dmin/Dmax : ratio of the minimum to the maximum distance between two spacecraft. This is the simplest criterion. Dmin/Dmean : ratio of the minimum distance between two spacecraft to the mean interspacecraft distance (also defined as the characteristic size). Smin/Smax : ratio of the minimum to the maximum surface of a triangular face of the tetrahedron. Smin/Smean : ratio of the minimum surface of a triangular face of the tetrahedron to the mean value of these 4 triangles. S/Sreg : ratio of the total surface of the tetrahedron (4 faces added) to the surface of a regular tetrahedron for which the inter-spacecraft distance is equal to the mean inter-spacecraft distance, i.e. the characteristic size. (V/Vreg + S/Sreg + D/Dmin)/3 : combination of 3 other quality parameters. (V/Vreg + S/Sreg + 1)/3 : combination of 2 other quality parameters + 1. This criterion (without the division by 3) has been defined by Vom Stein et al. (Réf. 4). Amin/Amax : ratio of the minimum to the maximum angle between 2 triangular faces of the tetrahedron. Amin/Amean : ratio of the minimum angle between 2 triangular faces of the tetrahedron to the mean value. V/Vreg : ratio of the volume of the tetrahedron to the volume of a regular tetrahedron for which the inter-spacecraft distance is equal to the mean inter-spacecraft distance. $(V/Vsphe)^{1/3}$: ratio of the volume of a tetrahedron to the volume of the sphere defined by the 4 points corresponding to the summits of the tetrahedron. This ratio is reduced by a constant value in order to get a normalization between 0 and 1. This constant value corresponds to the ratio between the volume of a regular tetrahedron and the volume of the corresponding sphere. $V/S^{3/2}$: ratio of the volume of the tetrahedron to its surface with the 3/2 exponent for the sake of homogeneity.

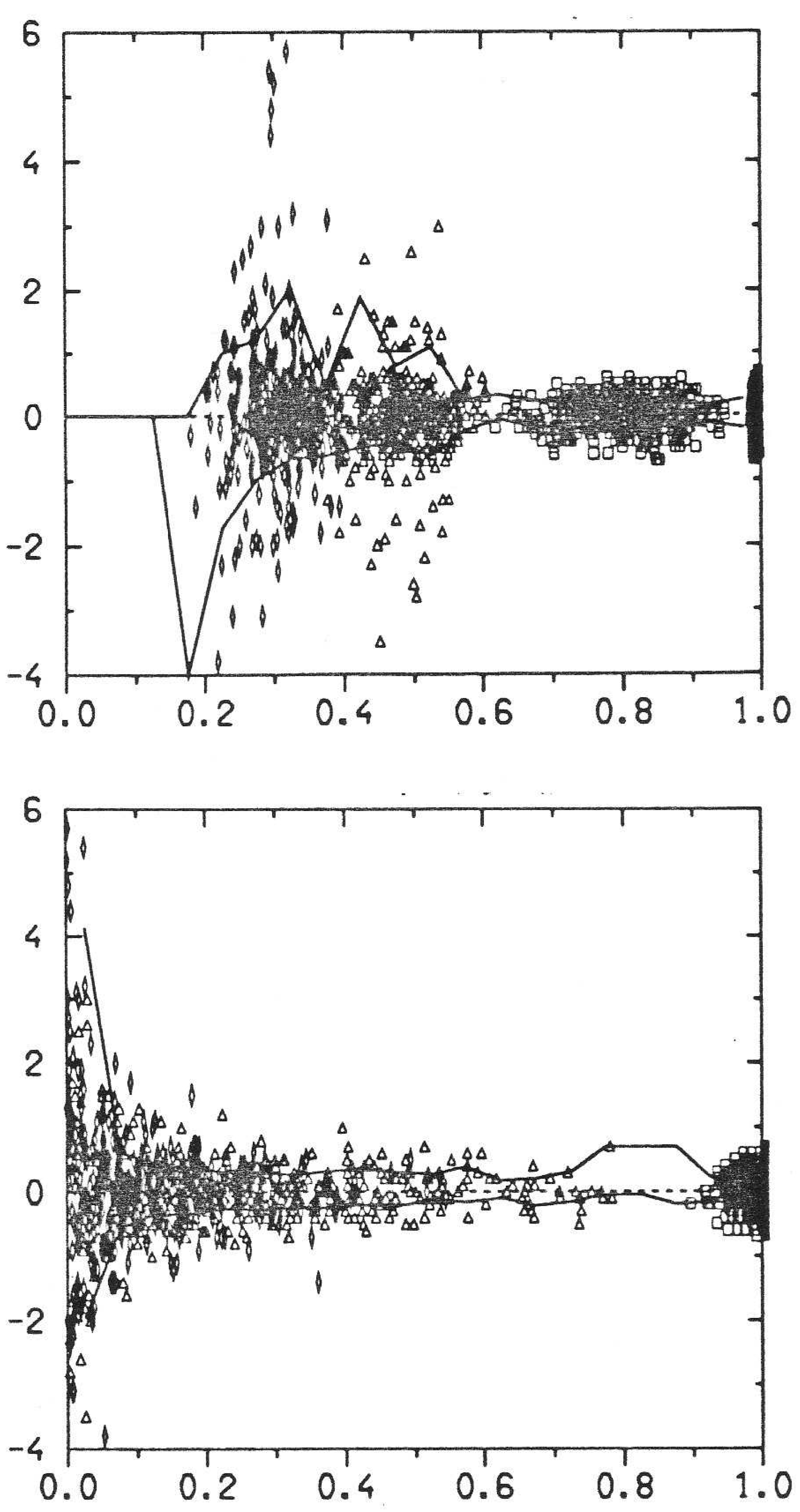

Fig. 4. Variation of the estimation of the current density with the value of the criteria Dmin/Dmax (top) and $(V/Vsphe)^{1/3}$ (bottom), for an homogeneous current density model. Uncertainty on distance measurement is fixed to a constant value $\Delta D/D = 1\%$.

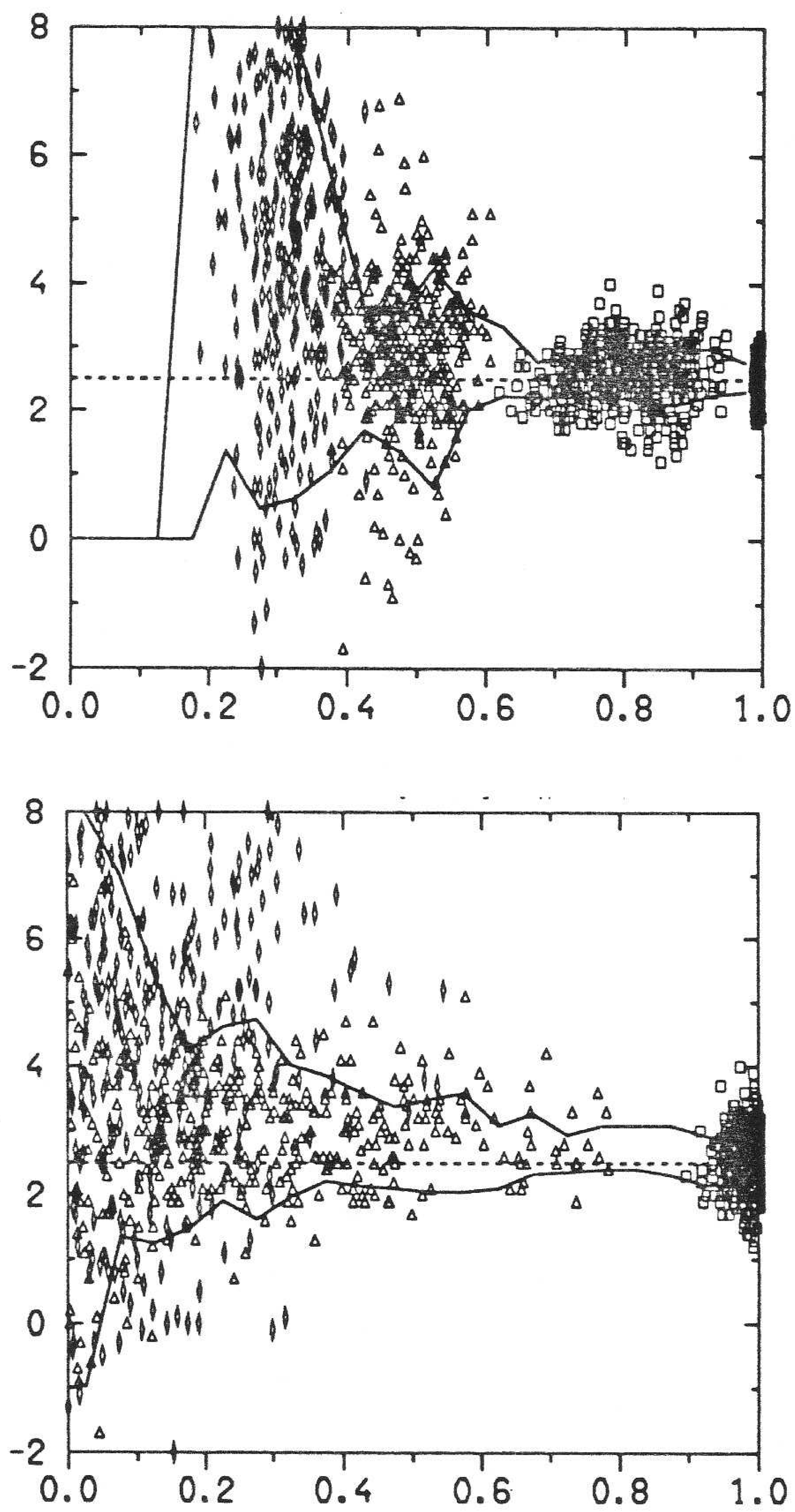

Fig. 5. Same as Fig. 4 but with a gaussian-shaped current density model with $Dc/\sigma = 0{,}5$ ($\sigma = 2000$ km).

For all these criteria, the values of $\Delta J/J$ are computed as described in section 2, but with a fixed relative uncertainty on $\Delta D/D$ taken to be 1%, which corresponds to the accuracy of the determination of Cluster spacecraft positions, and then plotted versus the quality parameter of the criterion under study. The results are shown in Figures 4 and 5, where the chosen criteria are Dmin/Dmax and $(V/Vsphe)^{1/3}$. Figure 4 corresponds to an homogeneous current model, and Figure 5 to a gaussian-shaped. For each of the 12 criteria studied, we have produced these kinds of plots. Because of lack of space, the figures corresponding to all tested criteria have not been produced, but copies can be obtained from the authors, upon request.

The results can roughly be split into two categories (as listed in Table 2):

(i) criteria that are very good at separating the various families of tetrahedra but give widely spread $\Delta J/J$; Criteria A to G belong to this category,

(ii) criteria that mix the various types of tetrahedra but lead to a clear partition of the value of $\Delta J/J$; Criteria H to L belong to this category.

Figure 4a shows $\Delta J/J$ versus the quality parameter corresponding to criteria A, in the case of an homogeneous current density profile. As indicated above, the various types or families of tetrahedron are well separated but the corresponding $\Delta J/J$ are widely spread, at least for Dmin/Dmax < 0.6. With this type of criterion, one can conclude that regular and pseudo-regular tetrahedra give good estimate of J, but one cannot know when pseudo-planar and pseudo-linear tetrahedra would give acceptable results. Criteria of the second category allow a quick assessment of the quality of the tetrahedron. Indeed, as illustrated in Figure 4b, when the quality parameter is less than say 0.2, $|\Delta J/J|$ remains below 1%. Thus, whatever the shape of the tetrahedron, when the quality factor is less than 0.2, one can be confident on the estimate of J, at least for an homogeneous current density.

When the current density profile is not homogeneous, the spreading of the points is much more important, whatever the criterion. Criterion of the first category (A to G) are still very good at partitioning the types of tetrahedra, as illustrated in Figure 5a. Yet, due to the enhanced spreading of the points, criteria belonging to the second category (H to L) lead to a less clear partition, as illustrated in Figure 5b. For both types of criteria, indeed, it seems to be sufficient to set the quality parameter above say 0.6 to get a sufficient accuracy, typically better than 2%. This number seems to be smaller than what is seen in the figure, but one has to remind that the systematic shift of the values of $\Delta J/J$ present in Figure 5b should not be taken into account, because it is not related to the uncertainty, as explained above.

Both Figures 5a and 5b confirm that regular and pseudo-regular tetrahedra lead to small $\Delta J/J$ and show that pseudo-linear tetrahedra are generally worse than pseudo-planar.

		indication on geometric shape	
1	A		
	B	Dmin/Dmax	good
	C	Dmin/Dmean	not as good
	D	Smin/Smax	bad
	E	Smin/Smean	good
	F	S/Sreg	not as good to bad
	G	(V/Vreg+S/Sreg+D/Dmin)/3	good
		(V/Vreg+S/Sreg+1)/3	not as good
		indication on accuracy measurement of $\Delta J/J$	
2	H	Amin/Amax	good
	I	Amin/Amean	not as good
	J	V/Vreg	good
	K	$(V/Vsphe)^{1/3}$	good
	L	$V/S^{3/2}$	not as good

Table 2. *The two main classes of criteria*

Table 2 summarizes the outcome of the tests of the 12 criteria described above, in the case of a non-homogeneous current density profile. Since the inhomogenity leads to an important dispersion of the points, the advantages of category 2 vis-à-vis category 1 are less clear. In this table, we have indicated some qualitative estimates about the ability of each criterion to sort out tetrahedra that lead to small uncertainties $\Delta J/J$. Should the reader want to make up his mind about the performance of each criterion, copies of the original colored figures produced in each case can be sent upon request.

4. CONCLUSION

To help planning of the Cluster mission, it would be very useful to define a simple geometric criterion, expressed as a quality parameter, allowing an easy statement of how accurate the estimate of the current density will be via 4-points measurements of **B**. We have proposed a method to test 12 possible criteria. In all cases, the regular and pseudo-regular tetrahedra lead to smaller values of $\Delta J/J$, as expected. Yet, in a number of cases, the pseudo-linear and pseudo-planar tetrahedra lead to small uncertainties ($\Delta J/J$). It is therefore useful to have a criteria that allows to extract easily the acceptable pseudo-linear and pseudo-planar configurations. Criteria belonging to the second category fulfill that goal; in particular criteria H and K give very good results. For a non-homogeneous profile, the advantage of category 2 over category 1 is less obvious but still exists.

5. REFERENCES

1. Balogh A, & al, in "The CLUSTER mission: Scientific and technical aspects of the instrument" October 1988, ESA Publications, Noordwijk, The Netherlands, *ESA SP-1103*.

2. Dunlop M W, Balogh A, Southwood D J, Elphic R C, Glassmeier K H & Neubauer F M 1990, Configurational sensitivity of multipoint magnetic field measurements, Proceedings of the International Workshop on Space Plasma Physics Investigations by Cluster and Regatta, *ESA-SP 306*, 23-29.

3. Robert P & Roux A 1990, Accuracy of the estimate of J via multipoint measurements, Proceedings of the International Workshop on Space Plasma Physics Investigations by Cluster and Regatta, *ESA-SP 306*, 29-35.

4. Vom Stein R, Glassmeier K H & Dunlop M, A configuration parameter for the Cluster satellites, Technical Report, to be published.